

STANDARD EMBEDDINGS OF SMOOTH SCHUBERT VARIETIES IN RATIONAL HOMOGENEOUS MANIFOLDS OF PICARD NUMBER 1

SHIN-YOUNG KIM AND KYEONG-DONG PARK

ABSTRACT. Schubert varieties are classical varieties and one of the important examples among spherical varieties. Especially, all smooth Schubert varieties in rational homogeneous manifolds of Picard number 1 are horospherical varieties. We characterize standard embeddings of smooth Schubert varieties in rational homogeneous manifolds of Picard number 1 by means of varieties of minimal rational tangents. In particular, we mainly consider nonhomogeneous smooth Schubert varieties in symplectic Grassmannians and in the 20-dimensional F_4 -homogeneous manifold of Picard number 1.

1. INTRODUCTION

A *rational homogeneous manifold* is a homogeneous space G/P for a complex simple Lie group G and a parabolic subgroup $P \subset G$. Under the action of a Borel subgroup B of G , the closure of a B -orbit in G/P is called a *Schubert variety* of G/P . For details about the parabolic subgroups and the Schubert varieties of G/P , see Springer [25].

Generic Schubert varieties are singular, and smooth Schubert varieties have been classified by using combinatorial and geometric methods (for the combinatorial smoothness criterion, see Billey-Postnikov [1]). Since parabolic subgroups of a simple Lie group are in one-to-one correspondence with subsets of the set of simple roots (i.e., the nodes of the corresponding Dynkin diagram), the Dynkin diagrams with a marking correspond to rational homogeneous manifolds of Picard number 1. A marked subdiagram of the marked Dynkin diagram of G/P defines a homogeneous submanifold G_0/P_0 of G/P , the G_0 -orbit of the base point $eP \in G/P$, then it is a smooth Schubert variety (see Section 2 of Hong-Mok [8]). Lakshmibai-Weyman [18] and Brion-Polo [2] showed that when G/P is a Hermitian symmetric space of compact type, any smooth Schubert variety in G/P is a homogeneous submanifold associated to a subdiagram of the marked Dynkin diagram of G/P . More generally, when G/P is associated to a long simple root, for smooth Schubert varieties it suffices to consider homogeneous submanifolds from Proposition 3.7 of Hong-Mok [8]. On the other hand, when G/P is associated to a short simple root, there is a smooth Schubert variety which is not homogeneous. Recently, Hong [3] and Hong-Kwon [5] have classified all smooth Schubert varieties in this case.

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A smooth Schubert variety Z of G/P is canonically embedded in G/P by an equivariant embedding induced from the inclusion $B \subset G$. By a *standard embedding* of Z into G/P , we will mean the composite of the canonical equivariant embedding and an element of the automorphism group of G/P . When G/P is associated to a simple root and a homogeneous submanifold G_0/P_0 is not linear, we have a characterization of standard embeddings of G_0/P_0 into G/P by means of varieties of minimal rational tangents as follows.

Theorem 1.1 (Theorem 1.2 of Hong-Mok [7], Theorem 1.2 of Hong-Park [9]). *Let X be a rational homogeneous manifold associated to a simple root and let Z be a nonlinear rational homogeneous manifold associated to a subdiagram of the marked Dynkin diagram of X . If f is a holomorphic embedding from a connected open subset U of Z into X which respects varieties of minimal rational tangents for a general point $z \in U$, then f extends to a standard embedding of Z into X .*

Given a uniruled projective variety X equipped with a minimal rational component \mathcal{K} , the *variety $\mathcal{C}_x(X) \subset \mathbb{P}(T_x X)$ of minimal rational tangents* (VMRT) at a general point $x \in X$ is defined by the closure of the space of the tangent vectors of minimal rational curves belonging to \mathcal{K} passing through x . For a general reference on the theory of rational curves and varieties of minimal rational tangents, see Kollár [17], Hwang-Mok [12], Hwang [11] and Mok [22].

For a holomorphic embedding $f: U \rightarrow X$ from an open subset U of a uniruled projective variety Z with a minimal rational component \mathcal{H} , we say that f *respects varieties of minimal rational tangents* if

$$df(\mathcal{C}_z(Z)) = df(\mathbb{P}(T_z Z)) \cap \mathcal{C}_{f(z)}(X)$$

for a general point $z \in U$, where $\mathcal{C}_z(Z)$ is the variety of minimal rational tangents of Z at $z \in Z$ associated to \mathcal{H} and $\mathcal{C}_{f(z)}(X)$ is the variety of minimal rational tangents of X at $f(z)$ associated to \mathcal{K} .

If X is a rational homogeneous manifold G/P associated to a simple root, then there is a canonical choice of a minimal rational component, namely, the irreducible family of lines \mathbb{P}^1 which are contained in X after we embed X into \mathbb{P}^N by the ample generator of the Picard group of X . Similarly, there is a canonical choice of a minimal rational component for a smooth Schubert variety. In this paper, we will prove a generalization of Theorem 1.1 in the case that Z is a smooth Schubert variety of G/P .

Theorem 1.2. *Let X be a rational homogeneous manifold associated to a simple root and let Z be a nonlinear smooth Schubert variety in X . If f is a holomorphic embedding from a connected open subset U of Z into X which respects varieties of minimal rational tangents for a general point $z \in U$, then f extends to a standard embedding of Z into X .*

We denote the rational homogeneous manifold G/P associated to a simple root α_i by (G, α_i) . Among rational homogeneous manifolds associated to a short simple root, since $(B_\ell, \alpha_\ell) \cong (D_{\ell+1}, \alpha_\ell)$, $(C_\ell, \alpha_1) \cong \mathbb{P}^{2\ell-1}$ and $(G_2, \alpha_1) \cong (B_3, \alpha_1) \cong \mathbb{Q}^5$ as complex manifolds, these three cases can be regarded as rational homogeneous manifolds associated to a long simple root. Moreover, any smooth Schubert variety in the 15-dimensional F_4 -homogeneous manifold (F_4, α_4) is a linear space by Theorem 1.3 of Hong-Kwon [5]. Thus, it suffices to consider nonhomogeneous

smooth Schubert varieties in the symplectic Grassmannians (Section 3) and in the 20-dimensional F_4 -homogeneous manifold (F_4, α_3) (Section 4).

2. HOROSPHERICAL VARIETIES AND CARTAN-FUBINI EXTENSION

2.1. Spherical and horospherical varieties. For a complex reductive algebraic group G , a complex algebraic variety with an action of G is called a G -variety. A G -spherical variety is a normal G -variety having an open orbit under the action of a Borel subgroup B of G . A normal G -variety is *horospherical* if G acts with an open orbit G/H isomorphic to a torus bundle over a rational homogeneous manifold, or equivalently, if the isotropy subgroup H of a general point contains the unipotent radical of a Borel subgroup B . Toric varieties and rational homogeneous manifolds are the well-known examples of horospherical varieties. Furthermore, by definition, Schubert varieties of a rational homogeneous manifold G/P are spherical.

Let $\{\alpha_1, \dots, \alpha_n\}$ be the system of simple roots of G and $P(\alpha_i)$ denote the maximal parabolic subgroup associated to a simple root α_i following the standard numbering (e.g. Humphreys [10]). For the corresponding system $\{\omega_1, \dots, \omega_n\}$ of fundamental weights, $V(\omega_i)$ denotes the irreducible G -representation space with the i -th fundamental weight ω_i as a highest weight. When we take a highest weight vector v_i of $V(\omega_i)$, the G -orbit of $[v_i]$ in $\mathbb{P}(V(\omega_i))$ is closed and isomorphic to a rational homogeneous manifold $G/P(\alpha_i)$ which is denoted by (G, α_i) .

If v_i and v_j are highest weight vectors of $V(\omega_i)$ and $V(\omega_j)$ respectively, we will consider the closure of G -orbit of the point $[v_i + v_j]$ in $\mathbb{P}(V(\omega_i) \oplus V(\omega_j))$. For any $i \neq j$, the open orbit $G \cdot [v_i + v_j]$ is isomorphic to the \mathbb{C}^* -bundle over a rational homogeneous manifold $G/(P(\alpha_i) \cap P(\alpha_j))$. Since the closure of $G \cdot [v_i + v_j]$ in $\mathbb{P}(V(\omega_i) \oplus V(\omega_j))$ is a normal variety by Proposition 2.1 of Hong [4], it is a horospherical G -variety and we denote it by (G, α_i, α_j) . All smooth projective horospherical varieties of Picard number 1 was classified by Pasquier [24] using the fact that any nonlinear smooth horospherical variety of Picard number 1 is of the form (G, α_i, α_j) .

Proposition 2.1 (Theorem 0.1 of Pasquier [24]). *Let G be a connected reductive algebraic group. A smooth projective horospherical G -variety X of Picard number 1 is either homogeneous or horospherical of rank 1. In the second case, its automorphism group $\text{Aut}(X)$ is a connected non-reductive linear algebraic group acting with exactly two orbits X_0 and Z ; moreover, X is uniquely determined by its two closed G -orbits $Y \subset X_0$ and Z , isomorphic to rational homogeneous manifolds G/P_Y and G/P_Z , respectively, where (G, P_Y, P_Z) is one of the triples of the following list:*

- (1) $(B_n, P(\alpha_{n-1}), P(\alpha_n))$ with $n \geq 3$;
- (2) $(B_3, P(\alpha_1), P(\alpha_3))$;
- (3) $(C_n, P(\alpha_k), P(\alpha_{k-1}))$ with $n \geq k \geq 2$;
- (4) $(F_4, P(\alpha_2), P(\alpha_3))$;
- (5) $(G_2, P(\alpha_2), P(\alpha_1))$.

Recently, Hong [4] showed that a smooth horospherical variety X of Picard number 1 can be embedded as a linear section into a rational homogeneous manifold of Picard number 1 except when X is $(B_n, \alpha_{n-1}, \alpha_n)$ for $n \geq 7$. For description of their tangent space based on weights and roots, see Proposition 2.6 of Kim [16].

Example 2.2 (Odd symplectic Grassmannian $(C_n, \alpha_k, \alpha_{k-1})$). Let V be a complex vector space endowed with a skew-symmetric bilinear form ω of maximal rank. We

denote the variety of all k -dimensional isotropic subspaces of V by $\text{Gr}_\omega(k, V) = \{W \subset V : \dim W = k, \omega|_W \equiv 0\}$. When $\dim V$ is even, say, $2n$, the form ω is a nondegenerate symplectic form and this variety $\text{Gr}_\omega(k, 2n)$ is the usual symplectic Grassmannian, which is homogeneous under the action of the symplectic group $\text{Sp}(2n)$. But when $\dim V$ is odd, say, $2n+1$, the skew-form ω has a one-dimensional kernel. The variety $\text{Gr}_\omega(k, 2n+1)$, called the *odd symplectic Grassmannian*, is not homogeneous and has two orbits under the action of its automorphism group if $2 \leq k \leq n$ (cf. Mihai [20] and Proposition 1.12 of Pasquier [24]). If $k = 1$, then the isotropic condition holds trivially so that $\text{Gr}_\omega(1, V)$ is just the linear space $\mathbb{P}^{\dim V - 1}$. Next, for $k = n+1$ the odd symplectic Grassmannian $\text{Gr}_\omega(n+1, 2n+1)$ is isomorphic to the symplectic Grassmannian $\text{Gr}_\omega(n, 2n)$ because any $(n+1)$ -dimensional isotropic subspace must contain the one-dimensional kernel of ω . In what follows we will assume that $2 \leq k \leq n$ when considering the odd symplectic Grassmannians.

Let S be an odd symplectic Grassmannian $\text{Gr}_\omega(k, 2n+1)$ for $2 \leq k \leq n$. Then S is a smooth Fano manifold of Picard number 1 and the automorphism group $\text{Aut}(S)$ of S is isomorphic to the semi-direct product $((\text{Sp}(2n) \times \mathbb{C}^*)/\{\pm 1\}) \ltimes \mathbb{C}^{2n}$. We know that S has two orbits under its automorphism group. The closed orbit $\{W \in \text{Gr}_\omega(k, 2n+1) : \text{Ker } \omega \subset W\}$ is isomorphic to the symplectic Grassmannian $\text{Gr}_\omega(k-1, 2n)$ and the open orbit $\{W \in \text{Gr}_\omega(k, 2n+1) : \text{Ker } \omega \not\subset W\}$ is isomorphic to the dual tautological bundle on the symplectic Grassmannian $\text{Gr}_\omega(k, 2n)$. In fact, considering the decomposition $\mathbb{C}^{2n+1} = \text{Ker } \omega \oplus \mathbb{C}^{2n}$, any $W \in \text{Gr}_\omega(k, 2n+1)$ containing $\text{Ker } \omega$ corresponds a point of $\text{Gr}_\omega(k-1, 2n)$. And the projection coming from the above decomposition gives a map from the open orbit onto $\text{Gr}_\omega(k, 2n)$ of which the fiber at a point $E \in \text{Gr}_\omega(k, 2n)$ is E^* (for details, see Proposition 4.3 of Mihai [20]). Consequently, the odd symplectic Grassmannian $\text{Gr}_\omega(k, 2n+1)$ has three orbits under the semisimple part $\text{Sp}(2n)$ of its automorphism group. In particular, the $\text{Sp}(2n)$ -closed orbit lying in the open orbit is isomorphic to a symplectic Grassmannian $\text{Gr}_\omega(k, 2n)$.

2.2. Second fundamental form and Cartan-Fubini extension. Let V be a finite-dimensional vector space and let $\mathcal{A} \subset \mathbb{P}(V)$ be a complex-analytic subvariety. Denote by $\tilde{\mathcal{A}} \subset V \setminus \{0\}$ the *affine cone* of \mathcal{A} , i.e., the pre-image $\pi^{-1}(\mathcal{A})$ of the canonical projection $\pi: V \setminus \{0\} \rightarrow \mathbb{P}(V)$. For a smooth point $\eta \in \tilde{\mathcal{A}}$, the *second fundamental form*

$$\sigma_\eta: T_\eta \tilde{\mathcal{A}} \times T_\eta \tilde{\mathcal{A}} \rightarrow V/T_\eta \tilde{\mathcal{A}}$$

of $\tilde{\mathcal{A}} \subset V$ at $\eta \in \tilde{\mathcal{A}}$ is defined by $\sigma_\eta(\xi, \zeta) = \nabla_\xi \hat{\zeta} \bmod T_\eta \tilde{\mathcal{A}}$ for any $\xi, \zeta \in T_\eta \tilde{\mathcal{A}}$, where $\hat{\zeta}$ is a local vector field with $\hat{\zeta}(\eta) = \zeta$, and ∇ is the Euclidean flat connection on the Euclidean space V . Another definition is given by the differential of the *Gauss map*

$$\Gamma: \tilde{\mathcal{A}} \rightarrow \text{Gr}(n, V), \quad \beta \in \tilde{\mathcal{A}} \mapsto [T_\beta \tilde{\mathcal{A}}] \in \text{Gr}(n, V)$$

at η , where $n = \dim \mathcal{A} + 1$. The differential of the Gauss map Γ at $\eta \in \tilde{\mathcal{A}}$ is a linear map

$$d\Gamma_\eta: T_\eta \tilde{\mathcal{A}} \rightarrow T_{[T_\eta \tilde{\mathcal{A}}]} \text{Gr}(n, V) \cong \text{Hom}(T_\eta \tilde{\mathcal{A}}, V/T_\eta \tilde{\mathcal{A}}).$$

Recall that the canonical isomorphism χ between the tangent space $T_{[W]} \text{Gr}(k, V)$ of a Grassmannian and $\text{Hom}(W, V/W)$ is given by $\xi \mapsto \chi_\xi$ with $\chi_\xi(w) := \rho'(0) + W$, where $\rho: D \rightarrow V$ is a *moving vector field* with $\rho(0) = w$ along a holomorphic path

$\hat{\xi}$ from a connected open subset $D \subset \mathbb{C}$ into $\text{Gr}(k, V)$ such that $\hat{\xi}(0) = W$ and $\hat{\xi}'(0) = \xi$. Here, $\rho: D \rightarrow V$ is called a moving vector field along a holomorphic path $\hat{\xi}$ if $\rho(t) \in \hat{\xi}(t)$ for every $t \in D$. If we use this canonical isomorphism, the differential of the Gauss map is described as follows. For $\xi, \zeta \in T_\eta \tilde{\mathcal{A}}$ we choose the following gadgets:

- a holomorphic path $\alpha: D \rightarrow \tilde{\mathcal{A}}$ with $\alpha(0) = \eta$ and $\alpha'(0) = \xi$,
- a vector field $\rho: D \rightarrow V$ along α with $\rho(0) = \zeta$, i.e., $\rho(t) \in T_{\alpha(t)} \tilde{\mathcal{A}}$ for every $t \in D$.

If we set $\hat{\xi} := \Gamma \circ \alpha: D \rightarrow \text{Gr}(n, V)$, then $\hat{\xi}(0) = [T_\eta \tilde{\mathcal{A}}]$ and $\hat{\xi}'(0) = d\Gamma_\eta(\xi) \in \text{Hom}(T_\eta \tilde{\mathcal{A}}, V/T_\eta \tilde{\mathcal{A}})$ under the canonical isomorphism. Since ρ is a moving vector field along $\hat{\xi}$,

$$d\Gamma_\eta(\xi)(\zeta) = \hat{\xi}'(0)(\zeta) = \rho'(0) + T_\eta \tilde{\mathcal{A}} \in V/T_\eta \tilde{\mathcal{A}}.$$

Restating the above construction, we have obtained a symmetric bilinear map, the second fundamental form, $\sigma_\eta(\alpha'(0), \rho(0)) = \rho'(0) + T_\eta \tilde{\mathcal{A}}$ for every holomorphic path α in $\tilde{\mathcal{A}}$ with $\alpha(0) = \eta$ and every moving vector field ρ along α .

For a subspace E of $T_\eta \tilde{\mathcal{A}}$ we define

$$\text{Ker } \sigma_\eta(\cdot, E) := \{\zeta \in T_\eta \tilde{\mathcal{A}} : \sigma_\eta(\zeta, \xi) = 0, \forall \xi \in E\}.$$

From the fact that $\tilde{\mathcal{A}}$ is a cone with the vertex at 0, it follows that $\sigma_\eta(\eta, \xi) = 0$ for any $\xi \in T_\eta \tilde{\mathcal{A}}$. In particular, $\mathbb{C}\eta$ is contained in $\text{Ker } \sigma_\eta(\cdot, E)$ for any subspace E of $T_\eta \tilde{\mathcal{A}}$. At $[\eta] = \pi(\eta) \in \mathcal{A}$ the tangent space $T_{[\eta]} \mathcal{A}$ is given by $(T_\eta \tilde{\mathcal{A}}/\mathbb{C}\eta) \otimes (\mathbb{C}\eta)^*$. Thus the second fundamental form $\sigma_\eta: T_\eta \tilde{\mathcal{A}} \times T_\eta \tilde{\mathcal{A}} \rightarrow V/T_\eta \tilde{\mathcal{A}}$ of $\tilde{\mathcal{A}}$ at η induces the projective second fundamental form $\bar{\sigma}_{[\eta]}: T_{[\eta]} \mathcal{A} \times T_{[\eta]} \mathcal{A} \rightarrow T_{[\eta]} \mathbb{P}(V)/T_{[\eta]} \mathcal{A}$ of \mathcal{A} at $[\eta]$. From now on we will use the notation $\sigma_{[\eta]}$ instead of $\bar{\sigma}_{[\eta]}$ for the sake of convenience. For a subspace \bar{E} of $T_{[\eta]} \mathcal{A}$ we define $\text{Ker } \sigma_{[\eta]}(\cdot, \bar{E})$ by $\{\bar{\zeta} \in T_{[\eta]} \mathcal{A} : \sigma_{[\eta]}(\bar{\zeta}, \bar{\xi}) = 0, \forall \bar{\xi} \in \bar{E}\}$.

Definition 2.3. Let (X, \mathcal{K}) and (Z, \mathcal{H}) be two polarized uniruled projective manifolds equipped with a minimal rational component. Assume that a holomorphic embedding $f: U \rightarrow X$ respects varieties of minimal rational tangents. We say that f is *nondegenerate with respect to $(\mathcal{K}, \mathcal{H})$* if

- its image $f(U)$ is not contained in the bad locus of \mathcal{K} and
- for a general point $z \in U$ and a general smooth point $\alpha \in \tilde{\mathcal{C}}_z(Z)$, $df(\alpha)$ is a smooth point of $\tilde{\mathcal{C}}_{f(z)}(X)$ and $\text{Ker } \sigma_{df(\alpha)}(\cdot, T_{df(\alpha)}(df(\tilde{\mathcal{C}}_z(Z))))$ is equal to $\mathbb{C}df(\alpha)$, where $\sigma_{df(\alpha)}$ denotes the second fundamental form of the affine cone $\tilde{\mathcal{C}}_{f(z)}(X) \subset T_{f(z)}X$ at $df(\alpha)$.

As the main ingredient for the proof of Theorem 1.2, we state the non-equidimensional Cartan-Fubini type extension theorem, which says the rational extension of germs of holomorphic maps respecting varieties of minimal rational tangents. For an analytic continuation along minimal rational curves and Cartan-Fubini extension, see also Mok [23].

Proposition 2.4 (Theorem 1.1 of Hong-Mok [7]). *Let (X, \mathcal{K}) and (Z, \mathcal{H}) be two uniruled projective manifolds equipped with a minimal rational component. Assume that Z is of Picard number 1 and that $\mathcal{C}_z(Z)$ is positive-dimensional at a general point $z \in Z$. Let $f: U \rightarrow X$ be a holomorphic immersion defined on a connected*

open subset $U \subset Z$. If f respects varieties of minimal rational tangents and is nondegenerate with respect to $(\mathcal{K}, \mathcal{H})$, then f extends to a rational map $F: Z \rightarrow X$.

To use this result, we need to compute the second fundamental form of the variety of minimal rational tangents as subvariety in the tangent space and to check the nondegeneracy of the pair of varieties of minimal rational tangents. Then we can apply the non-equidimensional Cartan-Fubini type extension theorem and get a rational extension $F: Z \rightarrow X$ of f . Up to the action of $\text{Aut}(X)$, $F(x_0) = x_0$ and $\mathcal{C}_{x_0}(F(Z)) = \mathcal{C}_{x_0}(Z)$ for a fixed general point $x_0 \in U \subset Z$. Since f sends minimal rational curves in Z to minimal rational curves in X and the tangency property of the two VMRTs of Z and $F(Z)$ at an intersection point does imply equality of these VMRTs in the case of smooth Schubert varieties in a rational homogeneous manifold of Picard number 1, we can extend the map inductively along minimal rational curves. Consequently, F is the identity map up to the action of $\text{Aut}(X)$.

3. SMOOTH SCHUBERT VARIETIES IN SYMPLECTIC GRASSMANNIANS

Let G be a connected simple Lie group of type C_ℓ and let X be a rational homogeneous manifold G/P associated to a simple root α_k ($1 \leq k \leq \ell$). Then X is the *symplectic Grassmannian* $\text{Gr}_\omega(k, 2\ell)$ of isotropic k -subspaces of $V = \mathbb{C}^{2\ell}$ with respect to a *symplectic form* ω on $\mathbb{C}^{2\ell}$, where a symplectic form means a nondegenerate skew-symmetric bilinear form on V . Take a basis $\{e_1, \dots, e_{2\ell}\}$ of V such that $\omega(e_{\ell-i}, e_{\ell+i+1}) = -\omega(e_{\ell+i+1}, e_{\ell-i}) = 1$ for $0 \leq i \leq \ell-1$, and all other $\omega(e_i, e_j)$ are zero. Define $F_j \subset V$ by the subspace generated by e_1, \dots, e_j for $1 \leq j \leq 2\ell$ and set $F_0 = \{0\}$. Then $F_{\ell-i}^\perp = F_{\ell+i}$ for $0 \leq i \leq \ell$. The symplectic group $G = \text{Sp}(V)$ naturally acts on $\text{Gr}_\omega(k, 2\ell)$ and the parabolic subgroup P is the isotropy subgroup of G at $[F_k]$. If $k = 1$, then the isotropic condition holds trivially so that $\text{Gr}_\omega(1, 2\ell)$ is just the linear space $\mathbb{P}^{2\ell-1}$. On the other hand, a rational homogeneous manifold associated to a long simple root α_ℓ is the Lagrangian Grassmannian $\text{Gr}_\omega(\ell, 2\ell)$ of which any smooth Schubert variety is a homogeneous submanifold associated to a subdiagram of the marked Dynkin diagram of $\text{Gr}_\omega(\ell, 2\ell)$ by Lakshmibai-Weyman [18], Brion-Polo [2] and Hong-Mok [8]. In what follows we will assume that $1 < k < \ell$.

Fix a k -dimensional isotropic subspace $E \subset V$. Since we can view X as a subvariety of the Grassmannian $\text{Gr}(k, V)$ of k -dimensional subspaces in V and the tangent space of a Grassmannian $\text{Gr}(k, V)$ at $[E]$ is naturally isomorphic to $\text{Hom}(E, V/E) = E^* \otimes V/E$, we have

$$T_{[E]}X = \{h \in \text{Hom}(E, V/E) : \omega(h(e_1), e_2) + \omega(e_1, h(e_2)) = 0, \forall e_1, e_2 \in E\}.$$

Putting $E^\perp := \{v \in V : \omega(v, e) = 0, \forall e \in E\}$, E^\perp is a subspace of dimension $2\ell - k$ containing E because E is an isotropic subspace. From the nondegeneracy of ω , the isomorphism $V/E^\perp \cong E^*$ is induced by the symplectic form ω . Then, under the map $\psi: E^* \otimes V/E \rightarrow E^* \otimes V/E^\perp \cong E^* \otimes E^*$, the tangent space of a symplectic Grassmannian $\text{Gr}_\omega(k, 2\ell)$ is the inverse image $\psi^{-1}(S^2 E^*)$ of the symmetric square $S^2 E^* \subset E^* \otimes E^*$ and can be identified with

$$T_{[E]} \text{Gr}_\omega(k, 2\ell) = (E^* \otimes (E^\perp/E)) \oplus S^2 E^*.$$

Minimal rational curves of X are lines of $\text{Gr}(k, 2\ell)$ lying on X . Thus, the variety $\mathcal{C}_{[E]}(X)$ of minimal rational tangents of X at a point $[E] \in X$ is the variety of

decomposable tensors in $T_{[E]}X$. If a decomposable tensor $h = e^* \otimes v$ is contained in $T_{[E]}X \subset E^* \otimes V/E$, then

$$\begin{aligned}\omega(v, e') &= \omega(h(e), e') = -\omega(e, h(e')) \\ &= -\omega(e, (e^* \otimes v)e') = -\omega(e, e^*(e')v) \\ &= -\omega(e, v)e^*(e') \quad \text{for all } e' \in E,\end{aligned}$$

that is, $\omega(v, \cdot) \in \mathbb{C}e^*$. Conversely, if $\omega(v, \cdot) \in \mathbb{C}e^*$, then $e^* \otimes v$ is contained in $T_{[E]}X$. Therefore, the affine cone of $\mathcal{C}_{[E]}(X)$ is

$$\tilde{\mathcal{C}}_{[E]}(X) = \{e^* \otimes v \in E^* \otimes (V/E) : \omega(v, \cdot) \in \mathbb{C}e^*\} \setminus \{0\}.$$

By Proposition 3.2.1 of Hwang-Mok [15] or Corollary 5.5 of Landsberg-Manivel [19], the variety \mathcal{A} of minimal rational tangents of $\text{Gr}_\omega(k, 2\ell)$ at a point $[E] \in \text{Gr}_\omega(k, 2\ell)$ is the projectivization of the affine cone

$$\tilde{\mathcal{A}} = \{u \otimes q + cu^2 : u \in U, q \in Q, c \in \mathbb{C} \setminus \{0\}\} \subset (U \otimes Q) \oplus S^2U$$

where $U = E^*$ and $Q = E^\perp/E$. Under the projection $\mathcal{A} \rightarrow \mathbb{P}(U) = \mathbb{P}^{k-1}$ defined by $u \otimes q + cu^2 \mapsto u$, \mathcal{A} is a \mathbb{P}^{2m} -bundle over \mathbb{P}^{k-1} , where $m = \ell - k$.

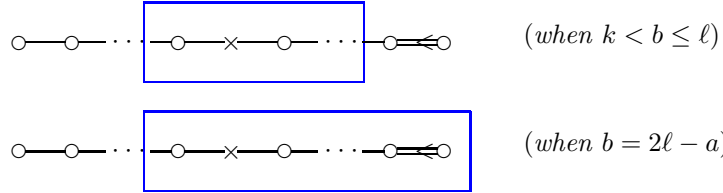
For integers a, b with $0 \leq a < k < b \leq 2\ell - a$, define

$$\text{Gr}_\omega(k, 2\ell; F_a, F_b) := \{E \in \text{Gr}_\omega(k, 2\ell) : F_a \subset E \subset F_b\},$$

where $F_j \subset V$ is the subspace generated by e_1, \dots, e_j . Recently, Hong [3] have classified smooth Schubert varieties in the symplectic Grassmannian $\text{Gr}_\omega(k, 2\ell)$. From this result, all smooth Schubert varieties are of this form satisfying a certain condition:

Lemma 3.1. *Smooth Schubert varieties of the symplectic Grassmannian $\text{Gr}_\omega(k, 2\ell)$ are of the form $\text{Gr}_\omega(k, 2\ell; F_a, F_b)$, where one of the following holds:*

- (1) $0 \leq a < k$ and $(k < b \leq \ell \text{ or } b = 2\ell - a)$; a homogeneous submanifold associated to a subdiagram of the marked Dynkin diagram corresponding to the symplectic Grassmannian $\text{Gr}_\omega(k, 2\ell)$,



- (2) $0 \leq a < k$ and $b = 2\ell - a - 1$; an odd symplectic Grassmannian $(C_{\ell-1}, \alpha_{k-a}, \alpha_{k-a-1})$,
- (3) $a = k - 1$ and $\ell + 1 \leq b \leq 2\ell - k$; a linear space \mathbb{P}^{b-k} .

Proof. Proposition 3.1 and Proposition 4.7 of Hong [3]. □

The odd symplectic Grassmannian $\text{Gr}_\omega(k, 2\ell; F_a, F_{2\ell-a-1})$ is not homogeneous but is a smooth Schubert variety of the symplectic Grassmannian $\text{Gr}_\omega(k, 2\ell)$ (see Example 2.2). Thus, for the proof of Theorem 1.2 in the case that X is the symplectic Grassmannian $\text{Gr}_\omega(k, 2\ell)$, it suffices to consider an odd symplectic Grassmannian (Case (2) of Lemma 3.1) for Z .

To prove Theorem 1.2, we will use the non-equidimensional Cartan-Fubini type extension theorem (Proposition 2.4). So we need to compute the second fundamental form of the variety of minimal rational tangents as subvariety in the tangent

space and to check the nondegeneracy of the pair of varieties of minimal rational tangents.

Lemma 3.2. *Let X be the symplectic Grassmannian $\text{Gr}_\omega(k, 2\ell)$ with $1 < k < \ell$ and \mathcal{A} be the variety of minimal rational tangents of X at a point $[E] \in X$. The tangent space T_β of $\tilde{\mathcal{A}}$ at $\beta \in \tilde{\mathcal{A}}$ is given by*

$$\begin{aligned} T_\beta &= \{u \otimes q' + u' \otimes q + 2u \circ u' : u' \in U, q' \in Q\} \text{ if } \beta = u \otimes q + u^2, \\ T_\beta &= \{u \otimes q' + u' \otimes q + cu^2 : u' \in U, q' \in Q, c \in \mathbb{C}\} \text{ if } \beta = u \otimes q. \end{aligned}$$

The second fundamental form $\sigma : T_\beta \times T_\beta \longrightarrow (T_{[E]}X)/T_\beta$ of $\tilde{\mathcal{A}} \subset T_{[E]}X$ at $\beta \in \tilde{\mathcal{A}}$ is given as follows:

$$\begin{aligned} \text{(a) for } \beta = u \otimes q + u^2, \\ \sigma(u' \otimes q + 2u \circ u', u \otimes q') &= u' \otimes q' \\ \sigma(u' \otimes q + 2u \circ u', u'' \otimes q + 2u \circ u'') &= 2u' \circ u'' \\ \sigma(u \otimes q', u \otimes q'') &= 0; \\ \text{(b) for } \beta = u \otimes q, \\ \sigma(u' \otimes q, u \otimes q') &= u' \otimes q' \\ \sigma(u' \otimes q, u^2) &= 2u \circ u' \\ \sigma(u' \otimes q, u'' \otimes q) &= 0 \\ \sigma(u \otimes q', u \otimes q'') &= 0 \\ \sigma(u \otimes q', u^2) &= 0 \\ \sigma(u^2, u^2) &= 0, \end{aligned}$$

where $u', u'' \in U$ and $q', q'' \in Q$.

Remark 3.3. The second fundamental form σ of $\tilde{\mathcal{A}}$ at $\beta \in \tilde{\mathcal{A}}$ has its image in the quotient space $(T_{[E]}X)/T_\beta$. For simplicity, here and henceforth we will use the same notation for an element $v \in T_{[E]}X$ and its image in the quotient $(T_{[E]}X)/T_\beta$. We will use the same convention for the second fundamental forms of other subvarieties.

Proof. This result is given in Lemma 3.2 of Hong-Park [9] without details. We give the details of the proof. First, to obtain the tangent space $T_\beta \tilde{\mathcal{A}}$, we consider the velocity vectors of curves in the affine cone $\tilde{\mathcal{A}}$. Let $\{u_t\} \subset U$ be a curve with $u_0 = u$ and $\{q_t\} \subset Q$ be a curve with $q_0 = q$. The curves $u_t \otimes q + u_t^2$, $u \otimes q_t + u^2$ in the affine cone $\tilde{\mathcal{A}}$ pass through a point $u \otimes q + u^2$ and their velocity vectors are $u' \otimes q + 2u \circ u'$ for some $u' \in U$ and $u \otimes q'$ for some $q' \in Q$, respectively. Because $\dim \mathcal{A} = k + 2m = \dim U + \dim Q$, the tangent space $T_\beta \tilde{\mathcal{A}}$ at a point $\beta = u \otimes q + u^2$ is spanned by the vectors $\{u' \otimes q + 2u \circ u' : u' \in U\}$ and $\{u \otimes q' : q' \in Q\}$. Similarly, the curves $u_t \otimes q$ and $u \otimes q_t$ pass through a point $u \otimes q$ when $t = 0$ so that their velocity vectors $\{u' \otimes q : u' \in U\}$ and $\{u \otimes q' : q' \in Q\}$ lie in $T_\beta \tilde{\mathcal{A}}$ at a point $\beta = u \otimes q$. But these vectors do not span the whole tangent space $T_\beta \tilde{\mathcal{A}}$ since $\{u' \otimes q : u' \in \mathbb{C}u\} = \{u \otimes q' : q' \in \mathbb{C}q\}$. Therefore, we additionally consider a curve $u \otimes q + c_t u^2$ such that $c_t \in \mathbb{C}$ and $c_0 = 0$, from which we obtain the tangent vectors of the form cu^2 for some $c \in \mathbb{C}$.

The second fundamental form $\sigma : T_\beta \times T_\beta \rightarrow (T_{[E]}X)/T_\beta$ is given by the differential of the Gauss map $\tilde{\mathcal{A}} \rightarrow \text{Gr}(n, T_{[E]}X)$, $\beta \mapsto [T_\beta \tilde{\mathcal{A}}]$, where $n = \dim \tilde{\mathcal{A}}$.

Let $\{u_t\} \subset U$ be a curve with $u_0 = u$ and $\{q_t\} \subset Q$ be a curve with $q_0 = q$. Then the holomorphic curves $[T_{\beta_t}]$ in $\text{Gr}(n, T_{[E]}X)$ for $\{\beta_t\} \subset \tilde{\mathcal{A}}$ such that $\beta_0 = \beta$ are as follows:

- (1) for $\beta_t = u_t \otimes q + u_t^2$, $T_{\beta_t} = \{u_t \otimes q' + u' \otimes q + 2u_t \circ u' : u' \in U, q' \in Q\}$;
- (2) for $\beta_t = u \otimes q_t + u^2$, $T_{\beta_t} = \{u \otimes q' + u' \otimes q_t + 2u \circ u' : u' \in U, q' \in Q\}$;
- (3) for $\beta_t = u_t \otimes q$, $T_{\beta_t} = \{u_t \otimes q' + u' \otimes q + cu_t^2 : u' \in U, q' \in Q, c \in \mathbb{C}\}$;
- (4) for $\beta_t = u \otimes q_t$, $T_{\beta_t} = \{u \otimes q' + u' \otimes q_t + cu^2 : u' \in U, q' \in Q, c \in \mathbb{C}\}$;
- (5) for $\beta_t = u \otimes q + c_t u^2$, $T_{\beta_t} = \{u \otimes q' + u' \otimes q + c_t u \circ u' + cu^2 : u' \in U, q' \in Q, c \in \mathbb{C}\}$, where $\{c_t\} \subset \mathbb{C}$ is a curve with $c_0 = 0$.

By differentiating the curve $[T_{\beta_t}]$ in $\text{Gr}(n, T_{[E]}X)$, we can compute the second fundamental form σ of $\tilde{\mathcal{A}}$. To be specific, for any tangent vectors $\xi, \zeta \in T_{\beta} \tilde{\mathcal{A}}$ we choose

- a holomorphic curve β_t into $\tilde{\mathcal{A}}$ such that $\beta_0 = \beta$ and $\frac{d}{dt}|_{t=0}\beta_t = \xi$, which gives the curve $[T_{\beta_t}]$ in $\text{Gr}(n, V)$,
- a vector field ρ_t along the above curve β_t such that $\rho_0 = \zeta$ and $\rho_t \in T_{\beta_t}$ for every t .

Then we have $\sigma(\xi, \zeta) = \sigma(\frac{d}{dt}|_{t=0}\beta_t, \rho_0) = \frac{d}{dt}|_{t=0}\rho_t$.

(Case I : $\beta = u \otimes q + u^2$). (i) First, to compute $\sigma(u' \otimes q + 2u \circ u', u \otimes q')$, take a curve $\beta_t = u_t \otimes q + u_t^2$ as in (1) and assume that $u_0 = u$, $\frac{d}{dt}|_{t=0}u_t = u'$. Then $\beta_0 = u \otimes q + u^2 = \beta$ and $\frac{d}{dt}|_{t=0}\beta_t = u' \otimes q + 2u \circ u'$. Since $u_t \otimes q' \in T_{\beta_t}$ for any t , the differential $\frac{d}{dt}|_{t=0}[T_{\beta_t}]: T_{\beta} \rightarrow T_{[E]}X/T_{\beta}$ maps $u \otimes q' \in T_{\beta}$ to $\frac{d}{dt}|_{t=0}u_t \otimes q' = u' \otimes q'$. Thus we have $\sigma(u' \otimes q + 2u \circ u', u \otimes q') = u' \otimes q'$.

(ii) Taking the same curve $\beta_t = u_t \otimes q + u_t^2$ as in (i), $u'' \otimes q + 2u_t \circ u'' \in T_{\beta_t}$ for any t . So $\sigma(u' \otimes q + 2u \circ u', u'' \otimes q + 2u \circ u'') = \frac{d}{dt}|_{t=0}(u'' \otimes q + 2u_t \circ u'') = 2u' \circ u''$.

(iii) For $\beta_t = u \otimes q_t + u^2$ with $\frac{d}{dt}|_{t=0}q_t = q'$ as in (2), $u \otimes q'' \in T_{\beta_t}$ for any t . So $\sigma(u \otimes q', u \otimes q'') = \frac{d}{dt}|_{t=0}u \otimes q'' = 0$.

(Case II : $\beta = u \otimes q$). (i) Similarly, we take $\beta_t = u_t \otimes q$ as in (3) and assume that $u_0 = u$, $\frac{d}{dt}|_{t=0}u_t = u'$. Then $\beta_0 = u \otimes q = \beta$, $\frac{d}{dt}|_{t=0}\beta_t = u' \otimes q$ and $u_t \otimes q' \in T_{\beta_t}$ for any t , hence $\sigma(u' \otimes q, u \otimes q') = \frac{d}{dt}|_{t=0}u_t \otimes q' = u' \otimes q'$.

(ii) For the above curve $\beta_t = u_t \otimes q$, $u_t^2 \in T_{\beta_t}$ for any t . So $\sigma(u' \otimes q, u^2) = \frac{d}{dt}|_{t=0}u_t^2 = 2u \circ u'$.

(iii) For the above curve $\beta_t = u_t \otimes q$, $u'' \otimes q \in T_{\beta_t}$ for any t . So $\sigma(u' \otimes q, u'' \otimes q) = \frac{d}{dt}|_{t=0}u'' \otimes q = 0$.

(iv) Taking $\beta_t = u \otimes q_t$ as in (4), $\beta_0 = u \otimes q = \beta$ and $\frac{d}{dt}|_{t=0}\beta_t = u \otimes q'$. Since $u \otimes q'' \in T_{\beta_t}$ for any t , we obtain $\sigma(u \otimes q', u \otimes q'') = \frac{d}{dt}|_{t=0}u \otimes q'' = 0$.

(v) For the above curve $\beta_t = u \otimes q_t$, $u^2 \in T_{\beta_t}$ for any t . So $\sigma(u \otimes q', u^2) = \frac{d}{dt}|_{t=0}u^2 = 0$.

(vi) Finally, we take $\beta_t = u \otimes q + c_t u^2$ as in (5) and assume that $\frac{d}{dt}|_{t=0}c_t = 1$. Then $\beta_0 = u \otimes q = \beta$ and $\frac{d}{dt}|_{t=0}\beta_t = u^2$. Since $u^2 \in T_{\beta_t}$ for any t , we obtain $\sigma(u^2, u^2) = \frac{d}{dt}|_{t=0}u^2 = 0$. \square

Definition 3.4. Let X be a polarized uniruled projective manifolds equipped with a minimal rational component \mathcal{K} . Assume that Z is an embedded submanifold in X . Let $\mathcal{A} := \mathcal{C}_x(X) \subset \mathbb{P}(T_x X)$ and $\mathcal{B} := \mathcal{C}_x(Z) \subset \mathbb{P}(T_x Z) \subset \mathbb{P}(T_x X)$ be the varieties of minimal rational tangents at a common general point x of X and Z ,

respectively. We say that the pair $(\mathcal{A}, \mathcal{B})$ is *nondegenerate* if

$$\text{Ker } \sigma_\beta(\cdot, T_\beta \tilde{\mathcal{B}}) = \mathbb{C}\beta$$

for any $\beta \in \tilde{\mathcal{B}}$, where $\sigma_\beta: T_\beta \tilde{\mathcal{A}} \times T_\beta \tilde{\mathcal{A}} \rightarrow (T_x X)/T_\beta \tilde{\mathcal{A}}$ is the second fundamental form of the affine cone $\tilde{\mathcal{A}}$ in $T_x X$ at β .

Proposition 3.5. *Let X be the symplectic Grassmannian $\text{Gr}_\omega(k, 2\ell)$ with $1 < k < \ell$ and Z be an odd symplectic Grassmannian $\text{Gr}_\omega(k, 2\ell; F_a, F_{2\ell-a-1})$ with $0 \leq a < k$. Let $\mathcal{A} \subset \mathbb{P}(T_x X)$ and $\mathcal{B} \subset \mathbb{P}(T_x Z) \subset \mathbb{P}(T_x X)$ be the varieties of minimal rational tangents at a common general point x of X and Z , respectively. Then the pair $(\mathcal{A}, \mathcal{B})$ is nondegenerate.*

Proof. Now take a point $[E] \in Z$ such that $E \cap F_{a+1} = F_a$. Then the dimension of $F_{2\ell-a-1} \cap E^\perp$ is $2\ell - k - 1$ and so $F_{2\ell-a-1}/(F_{2\ell-a-1} \cap E^\perp)$ is isomorphic to $(E/F_a)^*$. From Lemma 4.2 (2) of Hong-Mok [8], the variety \mathcal{B} of minimal rational tangent of Z at a general point $[E] \in Z$ is the projectivization of the affine cone

$$\tilde{\mathcal{B}} = \{u \otimes q + cu^2 : u \in U_a, q \in Q_a, c \in \mathbb{C}\} \setminus \{0\} \subset (U_a \otimes Q_a) \oplus S^2 U_a,$$

where $U_a = (E/F_a)^*$ and $Q_a = (F_{2\ell-a-1} \cap E^\perp)/E$. Note that Q_a is a codimension 1 subspace of Q . By this description of the varieties of minimal rational tangents of Z and by the computation of the second fundamental form of \mathcal{A} (Lemma 3.2), we get the desired results.

(i) The tangent space $T_\beta \tilde{\mathcal{B}}$ at $\beta = u \otimes q$ is given by $\{u \otimes q' + u' \otimes q + cu^2 : u' \in U_a, q' \in Q_a, c \in \mathbb{C}\}$. Then we have $\text{Ker } \sigma_\beta(\cdot, U_a \otimes q) = U \otimes q$, $\text{Ker } \sigma_\beta(\cdot, u \otimes Q_a) = \{u \otimes q' + cu^2 : q' \in Q, c \in \mathbb{C}\}$ and $\text{Ker } \sigma_\beta(\cdot, \mathbb{C}u^2) = \{u \otimes q' + cu^2 : q' \in Q, c \in \mathbb{C}\}$. Therefore, $\text{Ker } \sigma_\beta(\cdot, T_\beta \tilde{\mathcal{B}}) = \mathbb{C}(u \otimes q) = \mathbb{C}\beta$.

(ii) The tangent space $T_\beta \tilde{\mathcal{B}}$ at $\beta = u \otimes q + u^2$ is given by $\{u \otimes q' + u' \otimes q + 2u \circ u' : u' \in U_a, q' \in Q_a\}$. Then we have $\text{Ker } \sigma_\beta(\cdot, u \otimes Q_a) = \{u \otimes q' + cu^2 : q' \in Q, c \in \mathbb{C}\}$ and $\text{Ker } \sigma_\beta(\cdot, \{u' \otimes q + 2u \circ u' : u' \in U_a\}) = \mathbb{C}(u \otimes q + u^2)$. Therefore, $\text{Ker } \sigma_\beta(\cdot, T_\beta \tilde{\mathcal{B}}) = \mathbb{C}(u \otimes q + u^2) = \mathbb{C}\beta$. \square

We will use the same notation for $g \in G$ and for the differential of the action $g: X \rightarrow X$ at $x \in X$ and its projectivization $\mathbb{P}(T_x X) \rightarrow \mathbb{P}(T_{gx} X)$, for simplicity.

Proposition 3.6. *In the setting of Proposition 3.5, if $\mathcal{B}' = \mathcal{A} \cap \mathbb{P}(W')$ is another linear section of \mathcal{A} by a linear subspace $\mathbb{P}(W')$ of $\mathbb{P}(T_x X)$ such that $(\mathcal{B} \subset \mathbb{P}(T_x Z))$ is projectively equivalent to $(\mathcal{B}' \subset \mathbb{P}(W'))$, then there is an element h in the reductive part of the parabolic subgroup P such that $\mathcal{B}' = h\mathcal{B}$.*

Proof. Since \mathcal{B} is a \mathbb{P}^{2m-1} -bundle on $\mathbb{P}(U_a)$, \mathcal{B}' is also a \mathbb{P}^{2m-1} -bundle on \mathbb{P}^r , where $r = \dim U_a - 1 = k - a - 1$. Let \mathcal{B}'_1 be a codimension 1 linear section of \mathcal{B}' which is projectively equivalent to a codimension 1 linear section $\mathcal{B}_1 := \mathcal{B} \cap \mathbb{P}(U \otimes Q) \simeq \mathbb{P}(U_a) \times \mathbb{P}(Q_a)$ of \mathcal{B} .

Suppose that \mathcal{B}'_1 is not contained in $\mathbb{P}(U \otimes Q)$. Take $b' = u \otimes q + u^2 \in \tilde{\mathcal{B}}'_1 \cap (\tilde{\mathcal{A}} \setminus (U \otimes Q))$. Since \mathcal{B}' is a linear section $\mathcal{A} \cap \mathbb{P}(W')$ of \mathcal{A} , the tangent space $T_{b'} \tilde{\mathcal{B}}'$ at b' is contained in the intersection $W' \cap T_{b'} \tilde{\mathcal{A}}$ and the second fundamental form $\sigma_{b'}^{\mathcal{B}'} : T_{b'} \tilde{\mathcal{B}}' \times T_{b'} \tilde{\mathcal{B}}' \rightarrow W'/T_{b'} \tilde{\mathcal{B}}'$ of $\tilde{\mathcal{B}}'$ at b' , composed with the quotient map $W'/T_{b'} \tilde{\mathcal{B}}' \rightarrow T_x X/T_{b'} \tilde{\mathcal{A}}$, is the restriction of the second fundamental form σ of $\tilde{\mathcal{A}}$ to $T_{b'} \tilde{\mathcal{B}}' \times T_{b'} \tilde{\mathcal{B}}'$. Hence $\{v \in T_{b'} \tilde{\mathcal{B}}' : \sigma_{b'}^{\mathcal{B}'}(v, v) = 0\}$ is a linear subspace of $\{u \otimes q' : q' \in Q\}$ because \mathcal{B}' is a linear section of \mathcal{A} . Since Z is not linear, for

$b \in \tilde{\mathcal{B}}_1$, $\{v \in T_b \tilde{\mathcal{B}} : \sigma_b^{\mathcal{B}}(v, v) = 0\}$ is the union of two subspaces $\{u' \otimes q : u' \in U_a\}$ and $\{u \otimes q' : q' \in Q_a\}$, while for $b' \in \tilde{\mathcal{B}}'_1 \cap (\tilde{\mathcal{A}} \setminus (U \otimes Q))$, $\{v \in T_{b'} \tilde{\mathcal{B}}' : \sigma_{b'}^{\mathcal{B}'}(v, v) = 0\}$ is one linear subspace. Thus the second fundamental form $\sigma_b^{\mathcal{B}}$ is not isomorphic to $\sigma_{b'}^{\mathcal{B}'}$ and hence $\mathcal{B} \subset \mathbb{P}(T_x Z)$ cannot be projectively equivalent to $\mathcal{B}' \subset \mathbb{P}(W')$. Therefore, \mathcal{B}'_1 is contained in $\mathbb{P}(U \otimes Q) \cap \mathcal{A} \simeq \mathbb{P}(U) \times \mathbb{P}(Q)$. By Lemma 2 of Mok [21] about linear maps between nontrivial tensor product spaces, any linear section of $\mathbb{P}(U) \times \mathbb{P}(Q)$ which is projectively equivalent to $\mathbb{P}(U_a) \times \mathbb{P}(Q_a) \subset \mathbb{P}(T_x Z)$ is of the form $\mathbb{P}(U'_a) \times \mathbb{P}(Q'_a)$ for some subspaces $U'_a \subset U$ and $Q'_a \subset Q$ of $\dim U'_a = \dim U_a$, $\dim Q'_a = \dim Q_a$.

To characterize the variety \mathcal{B} of minimal rational tangents of Z , we use the *base locus* $\{v \in T_\beta \tilde{\mathcal{A}} : \sigma_\beta(v, v) = 0\}$ of the second fundamental form σ of $\tilde{\mathcal{A}} \subset V$ at a generic point $\beta \in \tilde{\mathcal{A}}$. Let $\mathcal{B}' = \mathcal{A} \cap \mathbb{P}(W')$ be a linear section of \mathcal{A} which is projectively equivalent to $\mathcal{B} \subset \mathbb{P}(W)$. Then for a general point β' of $\tilde{\mathcal{B}}'$ the second fundamental form $\sigma_{\beta'}^{\mathcal{B}'}$ of $\tilde{\mathcal{B}}'$ at β' is isomorphic to the second fundamental form $\sigma_\beta^{\mathcal{B}}$ of $\tilde{\mathcal{B}}$ at β . Hence $\{v \in T_{\beta'} \tilde{\mathcal{B}}' : \sigma_{\beta'}^{\mathcal{B}'}(v, v) = 0\}$ is isomorphic to $\{v \in T_\beta \tilde{\mathcal{B}} : \sigma_\beta^{\mathcal{B}}(v, v) = 0\}$. From the fact that \mathcal{B}' is a linear section of \mathcal{A} , it follows that $\{v \in T_{\beta'} \tilde{\mathcal{B}}' : \sigma_{\beta'}^{\mathcal{B}'}(v, v) = 0\}$ is contained in $\{v \in T_{\beta'} \tilde{\mathcal{A}} : \sigma_{\beta'}(v, v) = 0\}$. For $b \in \mathcal{B}' \setminus \mathcal{B}'_1 \subset \mathcal{A} \setminus \mathbb{P}(U \otimes Q)$ the linear space \mathbb{P}^{2m-1} in \mathcal{B}' passing through b is contained in the fiber of the projection $\mathcal{A} \rightarrow \mathbb{P}(U)$ containing b , because $\{v \in T_b \tilde{\mathcal{A}} : \sigma_b(v, v) = 0\}$ is the tangent space to the fiber of $\tilde{\mathcal{A}} \rightarrow U$. Thus \mathcal{B}' is the restriction of the \mathbb{P}^{2m-1} -bundle on $\mathbb{P}(U)$ to the subspace $\mathbb{P}(U'_a)$. Because any hyperplane in $\mathbb{P}(Q)$ can be transformed another hyperplane in $\mathbb{P}(Q)$ under the action of $\mathrm{Sp}(Q)$, $\mathcal{B}' = h\mathcal{B}$ for some $h \in \mathrm{SL}(U) \times \mathrm{Sp}(Q)$ which is the semisimple part of P . \square

Proof of Theorem 1.2 in the case that X is the symplectic Grassmannian $\mathrm{Gr}_\omega(k, 2\ell)$. From Lemma 3.1 and Theorem 1.1, it suffices to consider odd symplectic Grassmannians $\mathrm{Gr}_\omega(k, 2\ell; F_a, F_{2\ell-a-1})$ in the symplectic Grassmannian $\mathrm{Gr}_\omega(k, 2\ell)$ with $1 < k < \ell$. Let Z be an odd symplectic Grassmannian $\mathrm{Gr}_\omega(k, 2\ell; F_a, F_{2\ell-a-1})$ with $0 \leq a < k$.

Let $f: U \rightarrow X$ be a holomorphic embedding from a connected open subset U of Z into X which respects varieties of minimal rational tangents for a general point $z \in U$. Then $df(\mathcal{C}_z(Z))$ is the linear section $\mathcal{C}_{f(z)}(X) \cap df(\mathbb{P}(T_z Z))$ of $\mathcal{C}_{f(z)}(X)$ and $df(\mathcal{C}_z(Z)) \subset df(\mathbb{P}(T_z Z))$ is projectively equivalent to $\mathcal{C}_z(Z) \subset \mathbb{P}(T_z Z)$. By Proposition 3.6, for each general point $z \in U$ there is $g = g(z)$ in the reductive part of the parabolic subgroup P such that $df(\mathcal{C}_z(Z)) = \mathcal{C}_{f(z)}(gZ)$. Thus f is nondegenerate with respect to $(\mathcal{K}, \mathcal{H})$ by Proposition 3.5. Then Proposition 2.1 of Hong-Mok [7] implies that f preserves the tautological foliations, that is, f sends minimal rational curves in Z to minimal rational curves in X and we get a rational extension $F: Z \rightarrow X$ of f by Proposition 2.4. Then the total transformation $F(Z)$ of F is *rationally saturated*, i.e., for every smooth point $x \in F(Z)$ and for any minimal rational curve C on X passing through x , C must lie on $F(Z)$ whenever C is tangent to $F(Z)$ at x . For a general point x in $F(Z)$, the variety $\mathcal{C}_x(F(Z))$ of minimal rational tangents of $F(Z)$ is $dF(\mathcal{C}_z(Z))$ where $x = F(z)$.

Fix a general point $x_0 \in U$. From the homogeneity $F(x_0) = x_0$ up to the action of G and $\mathcal{C}_{x_0}(F(Z)) = \mathcal{C}_{x_0}(Z)$ up to the action of G by Proposition 3.6. Then $F(\Sigma) = \Sigma$, where Σ denotes the subvariety of Z swept out by minimal rational curves in Z passing through x_0 . Let C be a standard minimal rational curve in Z

passing through x_0 and let $y \in C$ be a smooth point different from x_0 . Then the tangent direction $[T_y C]$ is contained both in $\mathcal{C}_y(Z)$ and in $\mathcal{C}_y(F(Z))$. By the theory of minimal rational curves (Lemma 2.8 of Hong-Mok [7]), the tangent space $T_y \Sigma$ of Σ at x can be identified with the tangent space of $\tilde{\mathcal{C}}_y(Z)$ at $\alpha \in T_x C$. Note that by Proposition 4.3 of Hong-Mok [8], if h is an element in the isotropy subgroup $P_{[E]}$ of G such that $h\mathcal{B}$ and \mathcal{B} are tangent at a point of intersection, then $h\mathcal{B}$ is equal to \mathcal{B} . Since $F(\Sigma) = \Sigma$, $\mathcal{C}_y(Z)$ is tangent to $\mathcal{C}_y(F(Z))$ at $[T_y C]$, and thus we have $\mathcal{C}_y(F(Z)) = \mathcal{C}_y(Z)$. Therefore, $\mathcal{C}_y(F(Z)) = \mathcal{C}_y(Z)$ for a generic point $y \in \Sigma$.

Since Z is a uniruled projective manifold of Picard number 1, there is a sequence of irreducible varieties $\mathcal{U}_0 = \{x_0\} \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_k$ with $\dim \mathcal{U}_k = \dim Z$ such that a general point in \mathcal{U}_{i+1} can be connected to a point in \mathcal{U}_i by a minimal rational curve in Z . Applying the same arguments as above inductively, we get that $F(\mathcal{U}_k) = \mathcal{U}_k$ and thus we have $F(Z) = Z$. Consequently, F is the identity map up to the action of G . Therefore, f is the restriction of the standard embedding of Z into X . \square

4. SMOOTH SCHUBERT VARIETIES IN F_4 -HOMOGENEOUS MANIFOLDS

Let us start with the facts about the complex simple Lie algebra \mathfrak{g} of type F_4 . We choose a system $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of simple roots such that α_3 and α_4 are short roots. Then the highest long root of \mathfrak{g} is $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$, hence the grading on \mathfrak{g} associated to α_3 is of depth 4.

$$(F_4) \quad \begin{array}{ccccccc} & 2 & & 3 & & 4 & & 2 \\ & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\ & \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 \end{array}$$

Let \mathfrak{p} be the maximal parabolic subalgebra of \mathfrak{g} associated to the simple root α_3 . Given an integer k , $-4 \leq k \leq 4$, Φ_k denotes the set of all roots $\alpha = \sum_{q=1}^4 c_q \alpha_q$ with the third coefficient $c_3 = k$. Define

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha, \quad \mathfrak{g}_k = \bigoplus_{\alpha \in \Phi_k} \mathfrak{g}_\alpha, \quad k \neq 0.$$

Then the parabolic subalgebra \mathfrak{p} is decomposed as a graded Lie algebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \mathfrak{g}_4$ with

$$\dim \mathfrak{g}_0 = 12, \dim \mathfrak{g}_1 = 6, \dim \mathfrak{g}_2 = 9, \dim \mathfrak{g}_3 = 2, \dim \mathfrak{g}_4 = 3.$$

Let G be a connected simple Lie group of type F_4 and let X be a rational homogeneous manifold G/P associated to a short root α_3 . Then X is the space of G -invariant lines in the rational homogeneous manifold of type (F_4, α_4) which is a smooth hyperplane section of the (complex) Cayley plane $\mathbb{O}\mathbb{P}^2 = (E_6, \alpha_1)$ (cf. Section 6 of Landsberg-Manivel [19]).

Since $\dim \mathfrak{g} = 52$ and $\dim \mathfrak{p} = 32$, the rational homogeneous manifold X of type (F_4, α_3) is a projective variety of dimension 20. Let $o = eP$ be the base point of $X = G/P$. The tangent space $T_o(G/P)$ is canonically isomorphic to $\mathfrak{g}/\mathfrak{p}$ and so the Chern number of the tangent bundle $T_{G/P}$ is computed by $\sum_{\beta \in \Phi_1 \cup \cdots \cup \Phi_4} \beta(H_{\alpha_3})$

from the proof of Proposition 1 in Hwang-Mok [13]. Because

$$\sum_{\beta \in \Phi_1} \beta(H_{\alpha_3}) + \sum_{\beta \in \Phi_2} \beta(H_{\alpha_3}) + \sum_{\beta \in \Phi_3} \beta(H_{\alpha_3}) + \sum_{\beta \in \Phi_4} \beta(H_{\alpha_3}) = 1 + 3 + 1 + 2 = 7,$$

the first Chern class of X is $c_1(X) = 7L \in H^2(X, \mathbb{Z}) \cong H^1(X, \mathcal{O}_X^*) = \text{Pic}(X)$. Here L is the ample generator with $\text{Pic}(G/P) = \mathbb{Z}L$ and gives an embedding $G/P \subset \mathbb{P}^{272} = \mathbb{P}(V(\omega_3))$, where ω_3 is the third fundamental weight of G . Furthermore, G/P is covered by lines of \mathbb{P}^{272} and the Chow space \mathcal{K}_o consists of all lines passing through o , which are contained in G/P . Hence the tangent map $\tau_o: \mathcal{K}_o \rightarrow \mathcal{C}_o$ is an embedding with the variety \mathcal{C}_o of minimal rational tangents at o is 5-dimensional, because $c_1(G/P) = 7L$.

Now we take a choice of a Levi factor \mathfrak{k} of \mathfrak{p} . The semisimple part of \mathfrak{k} is isomorphic to $\mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. So a reductive subgroup $K \subset P$ with Lie algebra \mathfrak{k} is isogenous to $\mathbb{C}^* \times \text{SL}(3, \mathbb{C}) \times \text{SL}(2, \mathbb{C})$ as a complex Lie group. Under the identification $T_o(G/P) = \mathfrak{g}/\mathfrak{p}$, $\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-4}$ is the graded decomposition into irreducible K -modules. Fixing a 3-dimensional complex vector space E^* and a 2-dimensional complex vector space Q , we can check the following K -module isomorphisms:

$$\mathfrak{g}_{-1} = E^* \otimes Q, \quad \mathfrak{g}_{-2} = \wedge^2 E^* \otimes S^2 Q, \quad \mathfrak{g}_{-3} = Q, \quad \mathfrak{g}_{-4} = E^*.$$

In particular, one can determine the highest weight variety $\mathbb{P}W \subset \mathbb{P}\mathfrak{g}_{-1}$ consisting of highest weight vectors of the irreducible K -module \mathfrak{g}_{-1} . Because the highest weight variety $\mathbb{P}W \subset \mathbb{P}\mathfrak{g}_{-1}$ of X is a homogeneous manifold associated to the marked Dynkin diagram having markings corresponding to the simple roots α_2 and α_4 which are adjacent to α_3 in the Dynkin diagram of the semisimple part of P , we have $\mathbb{P}W = \mathbb{P}^2 \times \mathbb{P}^1 \subset \mathbb{P}^5$ embedded in the Segre embedding and its affine cone is contained in $\{e^* \otimes q \in E^* \otimes Q : e \in E, q \in Q\} \setminus \{0\}$.

$$(F_4, \alpha_3) \quad \begin{array}{c} \circ \text{---} \circ \text{---} \times \text{---} \circ \end{array} \longrightarrow \begin{array}{c} \circ \text{---} \times \end{array} \quad \times \quad \mathbb{P}W \subset \mathbb{P}\mathfrak{g}_{-1}$$

The variety $\mathcal{C}_o(X)$ of minimal rational tangents at the base point $o \in X$ contains the highest weight variety $\mathbb{P}W$ in $\mathbb{P}\mathfrak{g}_{-1}$ but $\mathcal{C}_o(X)$ is strictly bigger than $\mathbb{P}W$ since $\dim \mathcal{C}_o = 5$. Hence it must contain the highest weight variety $\mathcal{W}_2 \subset \mathbb{P}\mathfrak{g}_{-2}$ with respect to the K -action. The affine cone of \mathcal{W}_2 is contained in

$$\{(e_1^* \wedge e_2^*) \otimes q^2 \in \wedge^2 E^* \otimes S^2 Q : e_1, e_2 \in E, q \in Q\} \setminus \{0\}.$$

And \mathcal{C}_o is contained in $\mathbb{P}(\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2})$.

By Section 3 of Hwang-Mok [14] or Proposition 6.9 of Landsberg-Manivel [19], the variety \mathcal{A} of minimal rational tangents of X at $o \in X$ is the projectivization of the cone

$$\{e^* \otimes q + (e_1^* \wedge e_2^*) \otimes q^2 : e \wedge e_1 \wedge e_2 = 0, e, e_1, e_2 \in E, q \in Q\} \setminus \{0\}$$

in $(E^* \otimes Q) \oplus (\wedge^2 E^* \otimes S^2 Q)$, where E is a complex vector space of dimension 3 and Q is a complex vector space of dimension 2. Since $\wedge^2 E^*$ is isomorphic to E as $\text{SL}(E)$ -module, we will make a fixed choice of the identification. Then the affine cone $\tilde{\mathcal{A}}$ over \mathcal{A} is given by

$$\tilde{\mathcal{A}} = \{e^* \otimes q + f \otimes q^2 : \langle e^*, f \rangle = 0, e^* \in E^*, f \in E, q \in Q\} \setminus \{0\},$$

where $\langle e^*, f \rangle$ denotes the evaluation of e^* at f . Under the projection map $e^* \otimes q + f \otimes q^2 \mapsto q$, \mathcal{A} is a fiber bundle over $\mathbb{P}(Q) = \mathbb{P}^1$ with fibers which are isomorphic to a 4-dimensional quadric \mathbb{Q}^4 . In other words, \mathcal{A} is the Grassmannian bundle of 2-planes of the vector bundle \mathcal{E}^* on \mathbb{P}^1 , where \mathcal{E} is a vector bundle of rank 4 which splits as $\mathcal{O}(1)^3 \oplus \mathcal{O}$.

Lemma 4.1. *Let X be a rational homogeneous manifold of type (F_4, α_3) and \mathcal{A} be the variety of minimal rational tangents of X at a point $x \in X$. The tangent space T_β of $\tilde{\mathcal{A}}$ at $\beta \in \tilde{\mathcal{A}}$ is given by*

$$\begin{aligned} T_\beta &= \{e^* \otimes q' + e'^* \otimes q + f' \otimes q^2 + f \otimes (2q \circ q') \\ &\quad : \langle e'^*, f \rangle + \langle e^*, f' \rangle = 0, e'^* \in E^*, f' \in E, q' \in Q\} \text{ if } \beta = e^* \otimes q + f \otimes q^2, \\ T_\beta &= \{e^* \otimes q' + e'^* \otimes q + f' \otimes q^2 : \langle e^*, f' \rangle = 0, e'^* \in E^*, f' \in E, q' \in Q\} \\ &\quad \text{if } \beta = e^* \otimes q. \end{aligned}$$

The second fundamental form $\sigma: T_\beta \times T_\beta \rightarrow (T_x X)/T_\beta$ of $\tilde{\mathcal{A}} \subset T_x X$ at $\beta \in \tilde{\mathcal{A}}$ is given as follows:

(a) for $\beta = e^* \otimes q + f \otimes q^2$,

$$\begin{aligned} \sigma(e^* \otimes q' + f \otimes (2q \circ q'), e^* \otimes q'' + f \otimes (2q \circ q'')) &= f \otimes (2q' \circ q'') \\ \sigma(e^* \otimes q' + f \otimes (2q \circ q'), e'^* \otimes q) &= e'^* \otimes q' \\ \sigma(e^* \otimes q' + f \otimes (2q \circ q'), f' \otimes q^2) &= f' \otimes (2q \circ q') \\ \sigma(e'^* \otimes q, e''^* \otimes q) &= 0 \\ \sigma(e'^* \otimes q, f' \otimes q^2) &= (-\langle e'^*, f' \rangle e) \otimes q^2 \\ \sigma(f' \otimes q^2, f'' \otimes q^2) &= 0; \end{aligned}$$

(b) for $\beta = e^* \otimes q$,

$$\begin{aligned} \sigma(e^* \otimes q', e^* \otimes q'') &= 0 \\ \sigma(e^* \otimes q', e'^* \otimes q) &= e'^* \otimes q' \\ \sigma(e^* \otimes q', f' \otimes q^2) &= f' \otimes (2q \circ q') \\ \sigma(e'^* \otimes q, e''^* \otimes q) &= 0 \\ \sigma(e'^* \otimes q, f' \otimes q^2) &= (-\langle e'^*, f' \rangle e) \otimes q^2 \\ \sigma(f' \otimes q^2, f'' \otimes q^2) &= 0, \end{aligned}$$

where $e'^*, e''^* \in E^*, f', f'' \in E$ and $q', q'' \in Q$.

Proof. This is given in Lemma 4.2 of Hong-Park [9] without details. We give the details of the proof. First, to obtain the tangent space $T_\beta \tilde{\mathcal{A}}$, we consider the velocity vectors of curves in the affine cone $\tilde{\mathcal{A}}$. Let $\{e_t^*\} \subset E^*, \{f_t\} \subset E$ and $\{q_t\} \subset Q$ be curves with $e_0^* = e^*$ and $q_0 = q$. Assuming $\langle e^*, f \rangle = 0$, the curve $e^* \otimes q_t + f \otimes q_t^2$ lies in the affine cone $\tilde{\mathcal{A}}$ and passes through a point $e^* \otimes q + f \otimes q^2$. Since its velocity vector is $e^* \otimes q' + f \otimes (2q \circ q')$ for some $q' \in Q$, $e^* \otimes q' + f \otimes (2q \circ q') \in T_\beta \tilde{\mathcal{A}}$. If we take e_t^* and f_t satisfying $\langle e_t^*, f_t \rangle = 0$ and $f_0 = f$, then $e_t^* \otimes q + f_t \otimes q^2$ is a curve passing through a point $e^* \otimes q + f \otimes q^2$ in $\tilde{\mathcal{A}}$ and its velocity vector is $e'^* \otimes q + f' \otimes q^2$ for some $e'^* \in E^*, f' \in E$ such that $\langle e'^*, f \rangle + \langle e^*, f' \rangle = 0$. Next, for the curve $\beta_t = e^* \otimes q_t + f_t \otimes q_t^2$ with $f_0 = 0$ and $\langle e^*, f_t \rangle = 0$, $\beta_0 = e^* \otimes q$ and $\frac{d}{dt}|_{t=0} \beta_t = e^* \otimes (\frac{d}{dt}|_{t=0} q_t) + (\frac{d}{dt}|_{t=0} f_t) \otimes q^2 + f_0 \otimes (\frac{d}{dt}|_{t=0} q_t^2) = e^* \otimes q' + f' \otimes q^2$ for some $f' \in E, q' \in Q$ such that $\langle e^*, f' \rangle = 0$.

By a similar computation as in Lemma 3.2, we get the above results. Let $\{e_t^*\} \subset E^*, \{f_t\} \subset E$ and $\{q_t\} \subset Q$ be curves with $e_0^* = e^*$ and $q_0 = q$. Then the holomorphic curves $[T_{\beta_t}]$ in $\text{Gr}(n, T_x X)$ for $\{\beta_t\} \subset \tilde{\mathcal{A}}$ such that $\beta_0 = \beta$ are as follows:

- (1) for $\beta_t = e^* \otimes q_t + f \otimes q_t^2$, $T_{\beta_t} = \{e^* \otimes q' + e'^* \otimes q_t + f' \otimes q_t^2 + f \otimes (2q_t \circ q') : \langle e'^*, f \rangle + \langle e^*, f' \rangle = 0, e'^* \in E^*, f' \in E, q' \in Q\}$;
- (2) for $\beta_t = e_t^* \otimes q + f \otimes q^2$, $T_{\beta_t} = \{e_t^* \otimes q' + e'^* \otimes q + f' \otimes q^2 + f \otimes (2q \circ q') : \langle e'^*, f \rangle + \langle e_t^*, f' \rangle = 0, e'^* \in E^*, f' \in E, q' \in Q\}$;
- (3) for $\beta_t = e^* \otimes q + f_t \otimes q^2$ with $f_0 = f$, $T_{\beta_t} = \{e^* \otimes q' + e'^* \otimes q + f' \otimes q^2 + f_t \otimes (2q \circ q') : \langle e'^*, f_t \rangle + \langle e^*, f' \rangle = 0, e'^* \in E^*, f' \in E, q' \in Q\}$;
- (4) for $\beta_t = e^* \otimes q_t$, $T_{\beta_t} = \{e^* \otimes q' + e'^* \otimes q_t + f' \otimes q_t^2 : \langle e^*, f' \rangle = 0, e'^* \in E^*, f' \in E, q' \in Q\}$;
- (5) for $\beta_t = e_t^* \otimes q$, $T_{\beta_t} = \{e_t^* \otimes q' + e'^* \otimes q + f' \otimes q^2 : \langle e_t^*, f' \rangle = 0, e'^* \in E^*, f' \in E, q' \in Q\}$;
- (6) for $\beta_t = e^* \otimes q + f_t \otimes q^2$ with $f_0 = 0$, $T_{\beta_t} = \{e^* \otimes q' + e'^* \otimes q + f' \otimes q^2 + f_t \otimes (2q \circ q') : \langle e'^*, f_t \rangle + \langle e^*, f' \rangle = 0, e'^* \in E^*, f' \in E, q' \in Q\}$.

As in Lemma 3.2, the second fundamental form is computed in the following manner : $\sigma(\frac{d}{dt}|_{t=0}\beta_t, \rho_0) = \frac{d}{dt}|_{t=0}\rho_t$, where ρ_t is a vector field on V along the curve β_t such that $\rho_t \in T_{\beta_t}$ for every t .

(Case I : $\beta = e^* \otimes q + f \otimes q^2$). (i) We take a curve $\beta_t = e^* \otimes q_t + f \otimes q_t^2$ as in (1) and assume that $\frac{d}{dt}|_{t=0}q_t = q'$. Then $\beta_0 = e^* \otimes q + f \otimes q^2 = \beta$ and $\frac{d}{dt}|_{t=0}\beta_t = e^* \otimes q' + f \otimes (2q \circ q')$. Since $e^* \otimes q'' + f \otimes (2q_t \circ q'') \in T_{\beta_t}$ for any t , the differential $\frac{d}{dt}|_{t=0}[T_{\beta_t}] : T_\beta \rightarrow V/T_\beta$ maps $e^* \otimes q'' + f \otimes (2q \circ q'') \in T_\beta$ to $\frac{d}{dt}|_{t=0}(e^* \otimes q'' + f \otimes (2q_t \circ q'')) = f \otimes (2(\frac{d}{dt}|_{t=0}q_t) \circ q'') = f \otimes (2q' \circ q'')$. Thus we have $\sigma(e^* \otimes q' + f \otimes (2q \circ q'), e^* \otimes q'' + f \otimes (2q \circ q'')) = f \otimes (2q' \circ q'')$. (ii) If $e'^* \otimes q \in T_\beta$, then the relation $\langle e'^*, f \rangle = 0$ holds. For the above curve $\beta_t = e^* \otimes q_t + f \otimes q_t^2$, $e'^* \otimes q_t \in T_{\beta_t}$ for any t . So $\sigma(e^* \otimes q' + f \otimes (2q \circ q'), e'^* \otimes q) = \frac{d}{dt}|_{t=0}(e'^* \otimes q_t) = e'^* \otimes q'$. (iii) If $f' \otimes q^2 \in T_\beta$, then the relation $\langle e^*, f' \rangle = 0$ holds. For the above curve $\beta_t = e^* \otimes q_t + f \otimes q_t^2$, $f' \otimes q_t^2 \in T_{\beta_t}$ for any t . So $\sigma(e^* \otimes q' + f \otimes (2q \circ q'), f' \otimes q^2) = \frac{d}{dt}|_{t=0}(f' \otimes q_t^2) = f' \otimes (2q \circ q')$. (iv) Taking a curve $\beta_t = e_t^* \otimes q + f \otimes q^2$ as in (2) such that $\frac{d}{dt}|_{t=0}e_t^* = e'^*$, $\beta_0 = e^* \otimes q + f \otimes q^2 = \beta$ and $\frac{d}{dt}|_{t=0}\beta_t = e'^* \otimes q$. If $e''^* \otimes q \in T_\beta$, then the relation $\langle e''^*, f \rangle = 0$ holds. Since $e''^* \otimes q \in T_{\beta_t}$ for any t , we obtain $\sigma(e'^* \otimes q, e''^* \otimes q) = \frac{d}{dt}|_{t=0}e''^* \otimes q = 0$. (v) We take the above curve $\beta_t = e_t^* \otimes q + f \otimes q^2$ and a vector field $f_t \otimes q^2$ along β_t with $f_0 = f'$. Then $f_t \otimes q^2 \in T_{\beta_t}$ whenever $\langle e_t^*, f_t \rangle = 0$ for any t . Differentiating the equation $\langle e_t^*, f_t \rangle = 0$ at $t = 0$, we have $\langle e'^*, f' \rangle + \langle e^*, (\frac{d}{dt}|_{t=0}f_t) \rangle = 0$. Hence $\frac{d}{dt}|_{t=0}f_t = -\langle e'^*, f' \rangle e$ and so $\sigma(e'^* \otimes q, f' \otimes q^2) = \frac{d}{dt}|_{t=0}f_t \otimes q^2 = (-\langle e'^*, f' \rangle e) \otimes q^2$. (vi) Taking a curve $\beta_t = e^* \otimes q + f_t \otimes q^2$ as in (3) such that $f_0 = f$ and $\frac{d}{dt}|_{t=0}f_t = f'$, $\beta_0 = e^* \otimes q + f \otimes q^2 = \beta$ and $\frac{d}{dt}|_{t=0}\beta_t = f' \otimes q^2$. Since $f'' \otimes q^2 \in T_{\beta_t}$ for any t , we obtain $\sigma(f' \otimes q^2, f'' \otimes q^2) = \frac{d}{dt}|_{t=0}f'' \otimes q^2 = 0$.

(Case II : $\beta = e^* \otimes q$). (i) Now take a curve $\beta_t = e^* \otimes q_t$ as in (4) and assume that $\frac{d}{dt}|_{t=0}q_t = q'$. Then $\beta_0 = e^* \otimes q = \beta$ and $\frac{d}{dt}|_{t=0}\beta_t = e^* \otimes q'$. Since $e^* \otimes q'' \in T_{\beta_t}$ for any t , we have $\sigma(e^* \otimes q', e^* \otimes q'') = \frac{d}{dt}|_{t=0}e^* \otimes q'' = 0$. (ii) For the above curve $\beta_t = e^* \otimes q_t$, $e'^* \otimes q_t \in T_{\beta_t}$ for any t . So $\sigma(e^* \otimes q', e'^* \otimes q) = \frac{d}{dt}|_{t=0}(e'^* \otimes q_t) = e'^* \otimes q'$. (iii) If $f' \otimes q^2 \in T_\beta$, then the relation $\langle e^*, f' \rangle = 0$ holds. For the above curve $\beta_t = e^* \otimes q_t$, $f' \otimes q_t^2 \in T_{\beta_t}$ for any t . So $\sigma(e^* \otimes q', f' \otimes q^2) = \frac{d}{dt}|_{t=0}(f' \otimes q_t^2) = f' \otimes (2q \circ q')$.

- (iv) Taking a curve $\beta_t = e_t^* \otimes q$ as in (5) such that $\frac{d}{dt}|_{t=0} e_t^* = e'^*$, $\beta_0 = e^* \otimes q = \beta$ and $\frac{d}{dt}|_{t=0} \beta_t = e'^* \otimes q$. Since $e''^* \otimes q \in T_{\beta_t}$ for any t , we obtain $\sigma(e'^* \otimes q, e''^* \otimes q) = \frac{d}{dt}|_{t=0} e''^* \otimes q = 0$.
- (v) We take the above curve $\beta_t = e_t^* \otimes q$ and a vector field $f_t \otimes q^2$ along β_t with $f_0 = f'$. Then $f_t \otimes q^2 \in T_{\beta_t}$ whenever $\langle e_t^*, f_t \rangle = 0$ for any t . Differentiating this equation, we know $\frac{d}{dt}|_{t=0} f_t = -\langle e'^*, f' \rangle e$ as in (v) of Case I. Hence $\sigma(e'^* \otimes q, f' \otimes q^2) = \frac{d}{dt}|_{t=0} f_t \otimes q^2 = (-\langle e'^*, f' \rangle e) \otimes q^2$.
- (vi) Taking a curve $\beta_t = e^* \otimes q + f_t \otimes q^2$ as in (6) such that $f_0 = 0$ and $\frac{d}{dt}|_{t=0} f_t = f'$, $\beta_0 = e^* \otimes q = \beta$ and $\frac{d}{dt}|_{t=0} \beta_t = f' \otimes q^2$. Since $f'' \otimes q^2 \in T_{\beta_t}$ for any t , we obtain $\sigma(f' \otimes q^2, f'' \otimes q^2) = \frac{d}{dt}|_{t=0} f'' \otimes q^2 = 0$. \square

Hong-Kwon [5] have classified smooth Schubert varieties in the F_4 -homogeneous manifold (F_4, α_3) . Thus, for the proof of Theorem 1.2, it suffices to consider the only two cases for Z :

Lemma 4.2. *Let X be a rational homogeneous manifold of type (F_4, α_3) . A non-homogeneous smooth Schubert variety Z of X is one of the followings:*

- (a) *the horospherical variety $(B_3, \alpha_2, \alpha_3)$,*
- (b) *the horospherical variety $(C_2, \alpha_2, \alpha_1)$ which is isomorphic to a smooth Schubert variety $\text{Gr}_\omega(2, 6; F_0, F_5)$ in the symplectic Grassmannian (C_3, α_2) .*

Remark 4.3. Recall that all nonlinear homogeneous submanifolds associated to subdiagrams of the marked Dynkin diagram of (F_4, α_3) are (B_3, α_3) and (C_3, α_2) . As considered in Section 3, the odd symplectic Grassmannian $(C_2, \alpha_2, \alpha_1)$ is a smooth Schubert variety of (C_3, α_2) .



Lemma 4.4. *Let Z be a nonhomogeneous smooth Schubert variety of the rational homogeneous manifold of type (F_4, α_3) . Then the variety \mathcal{B} of minimal rational tangents of Z at a general point $z \in Z$ is*

- (a) *a \mathbb{P}^2 -bundle $\mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-2)^2)$ over $\mathbb{P}(Q) = \mathbb{P}^1$ if Z is of type $(B_3, \alpha_2, \alpha_3)$,*
- (b) *a \mathbb{P}^1 -bundle $\mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-2))$ over $\mathbb{P}(Q) = \mathbb{P}^1$ if Z is of type $(C_2, \alpha_2, \alpha_1)$.*

Proof. (a) Hong and Kim [6] showed that the variety of minimal rational tangents of the horospherical variety $(B_n, \alpha_{n-1}, \alpha_n)$ is $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-2}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(-2)^{n-1})$ by calculating the Chern number based on a gradation on its tangent space.

(b) As already described in Section 3, the variety of minimal rational tangents of the odd symplectic Grassmannian $(C_n, \alpha_k, \alpha_{k-1})$ is $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{k-1}}(-1)^{2n-2k+1} \oplus \mathcal{O}_{\mathbb{P}^{k-1}}(-2))$. \square

Let Z be a smooth Schubert variety of type $(B_3, \alpha_2, \alpha_3)$. The gradation on the tangent space of Z described in Proposition 25 of Kim [16] could be embedded in the gradation on the tangent space

$$\mathfrak{g}_{-1} = E^* \otimes Q, \quad \mathfrak{g}_{-2} = E \otimes S^2 Q, \quad \mathfrak{g}_{-3} = Q, \quad \mathfrak{g}_{-4} = E^*$$

as a linear section by Lemma 4.2 (a) after proper shifting of the gradation on the tangent space of Z . Let $\mathfrak{g}'_{-1} \oplus \mathfrak{g}'_{-2} \oplus \mathfrak{g}'_{-3}$ be the induced gradation on the tangent

space of Z from X . Then

$$\mathfrak{g}'_{-1} = F^* \otimes Q, \quad \mathfrak{g}'_{-2} = F^{*\perp} \otimes S^2 Q, \quad \mathfrak{g}'_{-3} = \wedge^2 F^{*\perp},$$

where $F^* \subset E^*$ is a fixed subspace of dimension 1 and $F^{*\perp} = \{f \in E : \langle e^*, f \rangle = 0, \forall e^* \in F^*\}$. Hence, the variety \mathcal{B} of minimal rational tangents of Z at a general point x is

$$\begin{aligned} \mathcal{B} &= \mathbb{P}(\{e^* \otimes q + f \otimes q^2 : e^* \in F^*, f \in F^{*\perp}, q \in Q\}) \text{ as a linear section of} \\ \mathcal{A} &= \mathbb{P}(\{e^* \otimes q + f \otimes q^2 : \langle e^*, f \rangle = 0, e^* \in E^*, f \in E, q \in Q\}). \end{aligned}$$

This \mathcal{B} is a \mathbb{P}^2 -bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^2)$ over $\mathbb{P}(Q) = \mathbb{P}^1$. This result is coincide with Lemma 4.4 (a).

Let Z be a smooth Schubert variety of type $(C_2, \alpha_2, \alpha_1)$. By Lemma 4.2 (b) and Proposition 25 of Kim [16], after proper shifting of the gradation on the tangent space of Z , we let $\mathfrak{g}'_{-1} \oplus \mathfrak{g}'_{-2}$ be the induced gradation on the tangent space of Z from X . Then

$$\mathfrak{g}'_{-1} = F^* \otimes Q, \quad \mathfrak{g}'_{-2} = F' \otimes S^2 Q,$$

where $F^* \subset E^*$ is the above fixed subspace and $F' \subset F^{*\perp}$ is an 1-dimensional subspace. Hence, the variety \mathcal{B} of minimal rational tangents of Z at a general point x is

$$\mathcal{B} = \mathbb{P}(\{e^* \otimes q + f \otimes q^2 : e^* \in F^*, f \in F', q \in Q\})$$

as a linear section of \mathcal{A} . This \mathcal{B} is a \mathbb{P}^1 -bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ over $\mathbb{P}(Q) = \mathbb{P}^1$. This result is coincide with Lemma 4.4 (b).

Proposition 4.5. *Let X be a rational homogeneous manifold of type (F_4, α_3) and Z be a smooth Schubert variety of type $(B_3, \alpha_2, \alpha_3)$ or $(C_2, \alpha_2, \alpha_1)$. Let $\mathcal{A} \subset \mathbb{P}(T_x X)$ and $\mathcal{B} \subset \mathbb{P}(T_x Z) \subset \mathbb{P}(T_x X)$ be the varieties of minimal rational tangents at a common general point x of X and Z . Then the pair $(\mathcal{A}, \mathcal{B})$ is nondegenerate.*

Proof. By this description of the variety \mathcal{B} of minimal rational tangents of Z as a linear section of the variety \mathcal{A} of minimal rational tangents of X and by the computation of the second fundamental form of \mathcal{A} , we get the desired results.

(1) Let Z be a smooth Schubert variety of type $(B_3, \alpha_2, \alpha_3)$. (i) The tangent space $T_\beta \tilde{\mathcal{B}}$ at $\beta = e^* \otimes q$ is given by $\{e^* \otimes q' + e'^* \otimes q + f' \otimes q^2 : e'^* \in F^*, f' \in F^{*\perp}, q' \in Q\}$. Then we have $\text{Ker } \sigma_\beta(\cdot, e^* \otimes Q) = e^* \otimes Q$, $\text{Ker } \sigma_\beta(\cdot, F^* \otimes q) = \{e'^* \otimes q + f' \otimes q^2 : e'^* \in E^*, f' \in F^{*\perp}\}$ and $\text{Ker } \sigma_\beta(\cdot, F^{*\perp} \otimes q^2) = \{e'^* \otimes q + f' \otimes q^2 : e'^* \in F^*, f' \in E\}$. Therefore, $\text{Ker } \sigma_\beta(\cdot, T_\beta \tilde{\mathcal{B}}) = \mathbb{C}(e^* \otimes q) = \mathbb{C}\beta$. (ii) The tangent space $T_\beta \tilde{\mathcal{B}}$ at $\beta = e^* \otimes q + f \otimes q^2$ is given by $\{e^* \otimes q' + e'^* \otimes q + f' \otimes q^2 + f \otimes (2q \circ q') : e'^* \in F^*, f' \in F^{*\perp}, q' \in Q\}$. Then we have $\text{Ker } \sigma_\beta(\cdot, F^* \otimes q) \cap \text{Ker } \sigma_\beta(\cdot, F^{*\perp} \otimes q^2) = \{e'^* \otimes q + f' \otimes q^2 : e'^* \in F^*, f' \in F^{*\perp}\}$ and $\text{Ker } \sigma_\beta(\cdot, \{e^* \otimes q' + 2f \otimes q \circ q' : q' \in Q\}) = \mathbb{C}(e^* \otimes q + f \otimes q^2)$. Therefore, $\text{Ker } \sigma_\beta(\cdot, T_\beta \tilde{\mathcal{B}}) = \mathbb{C}(e^* \otimes q + f \otimes q^2) = \mathbb{C}\beta$.

(2) Let Z be a smooth Schubert variety of type $(C_2, \alpha_2, \alpha_1)$. (i) The tangent space $T_\beta \tilde{\mathcal{B}}$ at $\beta = e^* \otimes q$ is given by $\{e^* \otimes q' + e'^* \otimes q + f' \otimes q^2 : e'^* \in F^*, f' \in F', q' \in Q\}$. Then we have $\text{Ker } \sigma_\beta(\cdot, e^* \otimes Q) = e^* \otimes Q$, $\text{Ker } \sigma_\beta(\cdot, F^* \otimes q) = \{e'^* \otimes q + f' \otimes q^2 : e'^* \in E^*, f' \in F'\}$ and $\text{Ker } \sigma_\beta(\cdot, F' \otimes q^2) = \{e'^* \otimes q + f' \otimes q^2 : e'^* \in F^*, f' \in E\}$. Therefore, $\text{Ker } \sigma_\beta(\cdot, T_\beta \tilde{\mathcal{B}}) = \mathbb{C}(e^* \otimes q) = \mathbb{C}\beta$. (ii) The tangent space $T_\beta \tilde{\mathcal{B}}$ at $\beta = e^* \otimes q + f \otimes q^2$ is given by $\{e^* \otimes q' + e'^* \otimes q + f' \otimes q^2 + f \otimes (2q \circ q') : e'^* \in F^*, f' \in F', q' \in Q\}$. Then we have $\text{Ker } \sigma_\beta(\cdot, F^* \otimes q) \cap \text{Ker } \sigma_\beta(\cdot, F' \otimes q^2) = \{e'^* \otimes q + f' \otimes q^2 : e'^* \in F^*, f' \in F^{*\perp}\}$

and $\text{Ker } \sigma_\beta(\cdot, \{e^* \otimes q' + 2f \otimes q \circ q' : q' \in Q\}) = \mathbb{C}(e^* \otimes q + f \otimes q^2)$. Therefore, $\text{Ker } \sigma_\beta(\cdot, T_\beta \tilde{\mathcal{B}}) = \mathbb{C}(e^* \otimes q + f \otimes q^2) = \mathcal{C}\beta$. \square

Proposition 4.6. *In the setting of Proposition 4.5, if h is an element in the isotropy subgroup P_x of G at a general point $x \in Z$ such that $h\mathcal{B}$ and \mathcal{B} are tangent at a general point of intersection, then $h\mathcal{B}$ is equal to \mathcal{B} .*

Proof. We recall $K = \mathbb{C}^* \times \text{SL}(E^*) \times \text{SL}(Q)$ -module isomorphisms:

$$\mathfrak{g}_{-1} = E^* \otimes Q, \quad \mathfrak{g}_{-2} = \wedge^2 E^* \otimes S^2 Q, \quad \mathfrak{g}_{-3} = Q, \quad \mathfrak{g}_{-4} = E^*$$

under the identification $T_o X \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-4}$ at the base point $o \in X = G/P$.

(1) Let Z be a smooth Schubert variety of type $(B_3, \alpha_2, \alpha_3)$. The variety of minimal rational tangents at a general point x is

$$\mathcal{B} = \mathbb{P}(\{e^* \otimes q + f \otimes q^2 : e^* \in F^*, f \in F^{*\perp}, q \in Q\}),$$

where $F^* \subset E^*$ is a subspace of dimension 1 and $F^{*\perp} = \{f \in E : \langle e^*, f \rangle = 0, \forall e^* \in F^*\}$.

Let h be an element in the isotropy subgroup P_x of G , then $h = h_0 h_1 h_2 h_3 h_4$ where $dh_i \in \mathfrak{g}_i$. The actions of h_i at $e^* \otimes q + f \otimes q^2 \in \mathcal{B}$ are

$$\begin{aligned} h_0.(e^* \otimes q + f \otimes q^2) &= h_0.e^* \otimes h_0.q + h_0.f \otimes h_0.q^2 \in E^* \otimes Q + \wedge^2 E^* \otimes S^2 Q \\ h_1.(e^* \otimes q + f \otimes q^2) &= h_1.(f \otimes q^2) \in E^* \otimes Q \\ h_i.(e^* \otimes q + f \otimes q^2) &= 0 \text{ for } i = 2, 3, 4. \end{aligned}$$

Since $h_1.(f \otimes q^2)$ is $h_1.f \otimes q$ or 0, it follows that

$$h.\mathcal{B} = \mathbb{P}(\{e^* \otimes q + f \otimes q^2 : e^* \in h_0.F^* + h_1.(F^{*\perp}), f \in h_0.(F^{*\perp}), q \in Q\}).$$

If \mathcal{B} and $h.\mathcal{B}$ intersect at a general point $\beta = e^* \otimes q + f \otimes q^2 \in \mathcal{B} \cap (h.\mathcal{B})$, then the tangent space $T_\beta(h.\tilde{\mathcal{B}})$ is given by

$$\{e^* \otimes q' + e'^* \otimes q + f' \otimes q^2 + f \otimes (2q \circ q') : e'^* \in h_0.F^* + h_1.(F^{*\perp}), f' \in h_0.(F^{*\perp}), q' \in Q\}.$$

By assumption, $T_\beta(h.\tilde{\mathcal{B}})$ coincide with $T_\beta(\tilde{\mathcal{B}})$, we see $h_0.F^* + h_1.(F^{*\perp}) = F^*$ and $h_0.(F^{*\perp}) = F^{*\perp}$. Hence, $h.\mathcal{B} = \mathcal{B}$.

(2) Let Z be a smooth Schubert variety of type $(C_2, \alpha_2, \alpha_1)$. The variety of minimal rational tangents at a general point x is

$$\mathcal{B} = \mathbb{P}(\{e^* \otimes q + f \otimes q^2 : e^* \in F^*, f \in F', q \in Q\}),$$

where $F^* \subset E^*$ and $F' \subset F^{*\perp}$ are subspaces of dimension 1.

If h is an element in the isotropy subgroup P_x of G , then

$$h.\mathcal{B} = \mathbb{P}(\{e^* \otimes q + f \otimes q^2 : e^* \in h_0.F^* + h_1.F', f \in h_0.F', q \in Q\}).$$

If \mathcal{B} and $h.\mathcal{B}$ intersect at a general point $\beta = e^* \otimes q + f \otimes q^2 \in \mathcal{B} \cap (h.\mathcal{B})$, then the tangent space $T_\beta(h.\tilde{\mathcal{B}})$ is given by

$$\{e^* \otimes q' + e'^* \otimes q + f' \otimes q^2 + f \otimes (2q \circ q') : e'^* \in h_0.F^* + h_1.F', f' \in h_0.F', q' \in Q\}.$$

By assumption, $T_\beta(h.\tilde{\mathcal{B}})$ coincides with $T_\beta(\tilde{\mathcal{B}})$, we see $h_0.F^* + h_1.F' = F^*$ and $h_0.(F^{*\perp}) = F^{*\perp}$. Hence, $h.\mathcal{B} = \mathcal{B}$. \square

Remark 4.7. In the proof (1) of Proposition 4.6, if $\dim h_0.(F^{*\perp}) = 1$, $\dim h_1.(F^{*\perp}) = 2$ and $h_0.F^* \subset h_1.(F^{*\perp})$, then $h.\mathcal{B} = \mathbb{P}(\{e^* \otimes q + f \otimes q^2 : e^* \in h_1.(F^{*\perp}), f \in h_0.(F^{*\perp}), q \in Q\})$ is isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$. Since $h\mathcal{B}$ and \mathcal{B} are tangent at a general point of intersection, it does not happen in its final.

Proposition 4.8. *In the setting of Proposition 4.5, if $\mathcal{B}' = \mathcal{A} \cap \mathbb{P}(W')$ is another linear section of \mathcal{A} by a linear subspace $\mathbb{P}(W')$ of $\mathbb{P}(T_x X)$ such that $(\mathcal{B} \subset \mathbb{P}(T_x Z))$ is projectively equivalent to $(\mathcal{B}' \subset \mathbb{P}(W'))$, then there is an element h in the reductive part of P such that $\mathcal{B}' = h\mathcal{B}$.*

Proof. Let Z be a smooth Schubert variety of type $(B_3, \alpha_2, \alpha_3)$. Since \mathcal{B} is a \mathbb{P}^2 -bundle on $\mathbb{P}(Q) = \mathbb{P}^1$, \mathcal{B}' is also a \mathbb{P}^2 -bundle on \mathbb{P}^1 . Let \mathcal{B}'_1 be a linear section of \mathcal{B}' which is projectively equivalent to the linear section $\mathcal{B}_1 := \mathcal{B} \cap \mathbb{P}(E^* \otimes Q) \simeq \mathbb{P}(F^*) \times \mathbb{P}(Q)$ of \mathcal{B} .

Suppose that \mathcal{B}'_1 is not contained in $\mathbb{P}(E^* \otimes Q)$. Take $b' = e^* \otimes q + f \otimes q^2 \in \tilde{\mathcal{B}}'_1 \cap (\tilde{\mathcal{A}} \setminus (E^* \otimes Q))$. Since \mathcal{B}' is a linear section $\mathcal{A} \cap \mathbb{P}(W')$ of \mathcal{A} , the tangent space $T_{b'}\tilde{\mathcal{B}}'$ at b' is contained in the intersection $W' \cap T_{b'}\tilde{\mathcal{A}}$ and the second fundamental form $\sigma_{b'}^{\mathcal{B}'} : T_{b'}\tilde{\mathcal{B}}' \times T_{b'}\tilde{\mathcal{B}}' \rightarrow W'/T_{b'}\tilde{\mathcal{B}}'$ of \mathcal{B}' at b' , composed with the quotient map $W'/T_{b'}\tilde{\mathcal{B}}' \rightarrow T_x X/T_{b'}\tilde{\mathcal{A}}$, is the restriction of the second fundamental form σ of $\tilde{\mathcal{A}}$ to $T_{b'}\tilde{\mathcal{B}}' \times T_{b'}\tilde{\mathcal{B}}'$. In particular, $\{v \in T_{b'}\tilde{\mathcal{B}}' : \sigma_{b'}^{\mathcal{B}'}(v, v) = 0\}$ is contained in

$$\begin{aligned} & \{v \in T_{b'}\tilde{\mathcal{A}} : \sigma_{b'}(v, v) = 0\} \\ &= \{e'^* \otimes q + f' \otimes q^2 : \langle e'^*, f \rangle + \langle e^*, f' \rangle = 0, \langle e'^*, f' \rangle = 0, e'^* \in E^*, f' \in E\}. \end{aligned}$$

Hence, $\{v \in T_{b'}\tilde{\mathcal{B}}' : \sigma_{b'}^{\mathcal{B}'}(v, v) = 0\}$ is a linear section of an affine cone of a hyperquadric \mathbb{Q}^3 because \mathcal{B}' is a linear section of \mathcal{A} . For $b \in \tilde{\mathcal{B}}_1$, $\{v \in T_b\tilde{\mathcal{B}} : \sigma_b^{\mathcal{B}}(v, v) = 0\}$ is the union of three subspaces $E'^* \otimes q$, $e^* \otimes Q$ and $F^{*\perp} \otimes q^2$, while for $b' \in \tilde{\mathcal{B}}'_1 \cap (\tilde{\mathcal{A}} \setminus (E^* \otimes Q))$, $\{v \in T_{b'}\tilde{\mathcal{B}}' : \sigma_{b'}^{\mathcal{B}'}(v, v) = 0\}$ is as above. Thus the second fundamental form $\sigma_b^{\mathcal{B}}$ is not isomorphic to $\sigma_{b'}^{\mathcal{B}'}$ and hence $\mathcal{B} \subset \mathbb{P}(T_x Z)$ cannot be projectively equivalent to $\mathcal{B}' \subset \mathbb{P}(W')$, which is a contradiction. Therefore, \mathcal{B}'_1 is contained in $\mathbb{P}(E^* \otimes Q) \cap \mathcal{A} \simeq \mathbb{P}(E^*) \times \mathbb{P}(Q)$. In particular, \mathcal{B}' has nonzero intersection $\mathcal{B}'_1 = \mathbb{P}(R^*) \times \mathbb{P}(Q)$ with $\mathbb{P}(E^*) \times \mathbb{P}(Q)$ for some subspaces $R^* \subset E^*$ of $\dim R^* = 1$.

Since \mathcal{B} and \mathcal{B}' is a linear section of \mathcal{A} and second fundamental forms of \mathcal{B} and \mathcal{B}' are equivalent, we see that $\{v \in T_{b'}\tilde{\mathcal{B}}' : \sigma_{b'}^{\mathcal{B}'}(v, v) = 0\}$ is contained in $\{v \in T_{b'}\tilde{\mathcal{A}} : \sigma_{b'}(v, v) = 0\}$ as a linear section;

$$\{e'^* \otimes q + f' \otimes q^2 : \langle e'^*, f \rangle + \langle e^*, f' \rangle = 0, e'^* \in R^*, f' \in R^{*\perp}\}.$$

More precisely, for $b' = e^* \otimes q + f \otimes q^2 \in \tilde{\mathcal{B}}' \setminus \tilde{\mathcal{B}}'_1 \subset \tilde{\mathcal{A}} \setminus (E^* \otimes Q)$, the space $\{v \in T_{b'}\tilde{\mathcal{B}}' : \sigma_{b'}(v, v) = 0\}$ contains $\{e'^* \otimes q + f' \otimes q^2 : \langle e'^*, f \rangle + \langle e^*, f' \rangle = 0, e'^* \in R^*, f' \in R^{*\perp}\}$, meanwhile, for $b = e^* \otimes q + f \otimes q^2 \in \tilde{\mathcal{B}} \setminus \tilde{\mathcal{B}}_1 \subset \tilde{\mathcal{A}} \setminus (E^* \otimes Q)$, the space $\{v \in T_b\tilde{\mathcal{B}} : \sigma_b(v, v) = 0\} = \{e'^* \otimes q + f' \otimes q^2 : \langle e'^*, f \rangle + \langle e^*, f' \rangle = 0, e'^* \in F^*, f' \in F^{*\perp}\}$. Because second fundamental forms of \mathcal{B}' and \mathcal{B} are equivalent, the dimensions of base locus are same, the space $\{v \in T_{b'}\tilde{\mathcal{B}}' : \sigma_{b'}(v, v) = 0\}$ should be $\{e'^* \otimes q + f' \otimes q^2 : \langle e'^*, f \rangle + \langle e^*, f' \rangle = 0, e'^* \in R^*, f' \in R^{*\perp}\}$ which is tangent to the fiber of the projection $\mathcal{A} \rightarrow \mathbb{P}(Q)$. Hence, $\mathcal{B}' = \mathbb{P}\{e^* \otimes q + f \otimes q^2 : q \in Q, e^* \in R^*, f \in R^{*\perp}\}$. Recall that F^* and R^* are linear subspaces in E^* the dimensions $\dim F^* = \dim R^* = 1$. Thus, $\mathcal{B}' = h\mathcal{B}$ for some $h \in \mathrm{SO}(E^*) \times \mathrm{SL}(Q)$, which is contained in the semisimple part of P .

For a smooth Schubert variety of type $(C_2, \alpha_2, \alpha_1)$, we can prove this in the same way. \square

Proof of Theorem 1.2 in the case that X is the F_4 -homogeneous manifold associated to a short root. Since any smooth Schubert variety in the 15-dimensional F_4 -homogeneous manifold (F_4, α_4) is a linear space by Theorem 1.3 of Hong-Kwon [5], it suffices to consider nonhomogeneous smooth Schubert varieties of type $(B_3, \alpha_2, \alpha_3)$ and $(C_2, \alpha_2, \alpha_1)$ in the 20-dimensional F_4 -homogeneous manifold (F_4, α_3) . Using Proposition 4.5, Proposition 4.6 and Proposition 4.8, the same argument in the proof of Theorem 1.2 given in Section 3 completes the proof. \square

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SHIN-YOUNG KIM, KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL 02455, KOREA

E-mail address: shinyoungkim@kias.re.kr

KYEONG-DONG PARK, CENTER FOR GEOMETRY AND PHYSICS, INSTITUTE FOR BASIC SCIENCE (IBS), POHANG 37673, KOREA

E-mail address: kdpark@ibs.re.kr